Gauge-invariant fields in the temporal gauge, Coulomb-gauge fields, and the Gribov ambiguity

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Abstract

We examine the relation between Coulomb-gauge fields and the gauge-invariant fields constructed in the temporal gauge for two-color QCD by comparing a variety of properties, including their equal-time commutation rules and those of their conjugate chromoelectric fields. We also express the temporal-gauge Hamiltonian in terms of gauge-invariant fields and show that it can be interpreted as a sum of the Coulomb-gauge Hamiltonian and another part that is important for determining the equations of motion of temporal-gauge fields, but that can never affect the time evolution of "physical" state vectors. We also discuss multiplicities of gauge-invariant temporal-gauge fields that belong to different topological sectors and that, in previous work, were shown to be based on the same underlying gauge-dependent temporal-gauge fields. We argue that these multiplicities of gauge-invariant fields are manifestations of the Gribov ambiguity. We show that the differential equation that bases the multiplicities of gauge-invariant fields on their underlying gauge-dependent temporal-gauge fields has nonlinearities identical to those of the "Gribov" equation, which demonstrates the non-uniqueness of Coulomb-gauge fields. These multiplicities of gauge-invariant fields — and, hence, Gribov copies — appear in the temporal gauge, but only with the imposition of Gauss's law and the implementation of gauge invariance; they do not arise when the theory is represented in terms of gauge-dependent fields and Gauss's law is left unimplemented.

1 Introduction

In earlier work, we have implemented the non-Abelian Gauss's law that applies in QCD by constructing states that are annihilated by the "Gauss's law operator" $\hat{\mathcal{G}}^a(\mathbf{r})$ for the temporal ($A_0^c = 0$) gauge, [1] where, for two-color QCD,

$$\mathcal{G}^a(\mathbf{r}) = \partial_j \Pi_j^a(\mathbf{r}) + g\epsilon^{abc} A_j^b(\mathbf{r})\Pi_j^c(\mathbf{r}),\;\; \hat{\mathcal{G}}^a(\mathbf{r}) = \mathcal{G}^a(\mathbf{r}) + j_0^a(\mathbf{r}),\;\; \text{and}\;\; j_0^a(\mathbf{r}) = g\,\psi^\dagger(\mathbf{r})\,\tfrac{\tau^a}{2}\,\psi(\mathbf{r}) \tag{1}$$

and where $\Pi_j^a(\mathbf{r})$ is the negative chromoelectric field as well as the momentum conjugate to the gauge field $A_j^a(\mathbf{r})$. We have, furthermore, used the gauge-invariant quark and gluon operator-valued fields constructed in Ref. [1] to transform the QCD Hamiltonian into a

form in which it is expressed in terms of these gauge-invariant fields. [2, 3] Most recently, we have studied the relation of the gauge-invariant to the gauge-dependent gauge fields in the temporal gauge. In particular, we have solved the nonlinear integral equation that expresses the requirement that the non-Abelian Gauss's law be implemented, and have discussed the consequences that the solutions of this integral equation have for the topology of the gauge-invariant gauge field.[4]

In this paper we will address a number of questions that pertain to the relation between QCD in the Coulomb gauge and our formulation in which QCD in the temporal (Weyl) gauge is expressed entirely in terms of gauge-invariant fields — *i.e.* in terms of operator-valued fields that commute with the generator displayed in Eq. (1). QCD in the Coulomb gauge has been discussed by Schwinger, [5, 6] by Christ and Lee, [7, 8] by Creutz *et. al.*, [9] and by Sakita and Gervais, [10] among others. Quantization of QCD in the Coulomb gauge encounters a number of difficulties: These include the well-known Gribov ambiguity that becomes an impediment when Gauss's law is inverted to solve for the timelike component of the gauge field. [11, 12] Attention has also been called to operator ordering problems encountered when noncommuting fields appear multiplicatively in the Coulomb-gauge QCD Hamiltonian. [7, 10] Some authors have circumvented the latter problem by treating the $A_0 = 0$ gauge fields as a set of Cartesian coordinates and the Coulomb-gauge fields as a set of curvilinear coordinates and using standard methods to transform from the former to the latter. [7, 9]

We will show in this work that the gauge-invariant fields we have constructed — the gauge fields as well as the chromoelectric field — have properties that so closely match those of the corresponding Coulomb-gauge fields, that they can be identified with them. The gauge-invariant gauge field we have constructed is transverse, as is the gauge field in the Coulomb gauge. The gauge-invariant fields obey commutation relations that are the same as those in the Coulomb gauge with the exception of operator order in the commutator. We will, in this work, construct a number of operators that are important in QCD dynamics — the Hamiltonian, the Gauss's law operator, the gluon and quark color-charge densities, etc. — so that they are expressed entirely in terms of these gauge-invariant fields (for short, we will call them the "gauge-invariant" operators). We will show that this gauge-invariant Hamiltonian is not the same as the Coulomb-gauge Hamiltonian, but that it contains a version of the latter accompanied by another term that is specific to the temporal gauge. This additional term in the gauge-invariant temporal-gauge Hamiltonian has no dynamical consequences in the subspace to which the physical state variables are restricted, so that the two operators — the Coulomb-gauge Hamiltonian and the gauge-invariant Hamiltonian — have identical physical consequences. However, the additional term in the gauge-invariant temporal-gauge Hamiltonian affects the equations of motion of the gauge and quark fields. Finally, we will also discuss the relation between the multiple solutions of the equations for the gauge-invariant fields that we obtained in Ref.[4] and the Gribov copies of the Coulomb-gauge fields.

The plan for this paper is as follows: In Section 2, we will compare the commutation rules for the Coulomb-gauge field with those for the gauge-invariant field that we constructed in the temporal gauge, and discuss the sense in which these two fields can be identified with each other. We will also make a similar comparison for the gauge-invariant momentum (and negative chromoelectric field). In this section, we will also construct a number of gauge-invariant operator-valued quantities, such as the gauge-invariant gluon and quark color-charge and color-current densities, and the gauge-invariant version of the Gauss's law operator given in Eq. (1). We will, furthermore, express the temporal-gauge Hamiltonian in terms of these gauge-invariant variables, and discuss its relation to the Coulomb-gauge Hamiltonian. In Section 3, we will discuss the multiple solutions of the nonlinear equation that determines the gauge-invariant field, and their relation to the Gribov copies of the Coulomb-gauge field. In Section 4, we will conclude with observations based on results presented in earlier sections.

2 Gauge-invariant temporal-gauge fields

The quantization of QCD in the temporal gauge avoids many of the problems encountered in quantizing QCD in the Coulomb gauge, but at the expense of leaving Gauss's law still to be implemented after the quantization has been carried out. In quantizing QCD, we make use of the Lagrangian for two-color QCD, in which the gauge fields are in the adjoint representation of SU(2):[1]

$$\mathcal{L} = -\frac{1}{4}F_{ij}^aF_{ij}^a + \frac{1}{2}F_{i0}^aF_{i0}^a + j_i^aA_i^a - j_0^aA_0^a + \mathcal{L}_{\rm gauge} - \bar{\psi}\left(m - i\gamma\cdot\partial\right)\psi \tag{2}$$

where

$$F_{ij}^a = \partial_jA_i^a - \partial_iA_j^a - g\epsilon^{abc}A_i^bA_j^c\,, \quad F_{i0}^a = \partial_0A_i^a + \partial_iA_0^a + g\epsilon^{abc}A_i^bA_0^c\,, \tag{3}$$

$$\text{with} \quad j_i^a = g\psi^\dagger\alpha_i\frac{\tau^a}{2}\psi, \quad \text{and} \quad j_0^a = g\psi^\dagger\frac{\tau^a}{2}\psi\,; \tag{4}$$

$\mathcal{L}_{\rm gauge}$ is a gauge-fixing term. When the gauge-fixing term $-A_0^aG^a$ is used in Eq. (2), and the Dirac-Bergmann method of constrained quantization is used,[13, 14] the Lagrange multiplier field G^a is incorporated into the time-derivative of Π_0^a, which, in this case is $D_i\Pi_i^a + j_0^a + G^a$. The presence of the Lagrange multiplier field in the secondary constraint $D_i\Pi_i^a + j_0^a + G^a = 0$ terminates the chain of secondary constraints very quickly, and leads to Dirac commutators that differ in only trivial ways from canonical Poisson commutators. However, the Gauss's law constraint is not imposed in that process. Alternatively, it is possible to entirely avoid the need to consider primary constraints in the temporal gauge

[1]we use nonrelativistic notation, in which all space-time indices are subscripted and designate contravariant components of contravariant quantities such as A_i^a or j_i^a, and covariant components of covariant quantities such as ∂_i. Repeated indices are summed from 1→3.

by using the gauge-fixing term $-\partial_0 A_0^a G^a$ instead of $-A_0^a G^a$, so that $-G^a$ becomes the momentum canonically conjugate to A_0. The gauge constraint then is $\partial_0 A_0 = 0$, which, with the imposition of $A_0 = 0$ and Gauss's law at one particular time, implements both, the gauge condition and Gauss's law for all times;[15] the same procedure can be extended to all axial gauges for which the gauge condition is $A_0^a + \gamma A_3^a = 0$, where γ is a variable real parameter.[16] Even in these cases, however, the implementation of Gauss's law at one time — at $t = 0$, for example — is still necessary. Finally, a very direct way of quantizing QCD in the temporal gauge is simply to set $A_0^a = 0$ in the original Lagrangian. Gauss's law is not one of the Euler-Lagrange equations in this formulation, and must be imposed after the basic quantization has been carried out.[17, 18, 19]

Ref. [1] addresses the imposition of Gauss's law in the temporal gauge by explicitly constructing the states that are annihilated by the non-Abelian Gauss's law operator given in Eq. (1). The mathematical apparatus required for that purpose also enables us to construct the gauge-invariant gauge and quark fields. This apparatus includes the defining equation for a nonlocal operator-valued functional of the gauge field — the so-called "resolvent field" $\overline{\mathcal{A}_i^\gamma}(\mathbf{r})$ — which has a central role in this construction. In the two-color SU(2) version of QCD, with which we are concerned in this work, the resolvent field appears in the gauge-invariant gluon field in the form

$$[\,A^b_{\mathsf{GI}\,i}(\mathbf{r})\,\tfrac{\tau^b}{2}\,] = V_{\mathcal{C}}(\mathbf{r})\,[\,A_i^b(\mathbf{r})\,\tfrac{\tau^b}{2}\,]\,V_{\mathcal{C}}^{-1}(\mathbf{r}) + \tfrac{i}{g}\,V_{\mathcal{C}}(\mathbf{r})\,\partial_i V_{\mathcal{C}}^{-1}(\mathbf{r}), \tag{5}$$

where $V_{\mathcal{C}}(\mathbf{r})$ incorporates the resolvent field, as shown by

$$V_{\mathcal{C}}(\mathbf{r}) = \exp\Big(-ig\overline{\mathcal{Y}^\alpha}(\mathbf{r})\tfrac{\tau^\alpha}{2}\Big)\,\exp\Big(-ig\mathcal{X}^\alpha(\mathbf{r})\tfrac{\tau^\alpha}{2}\Big) \tag{6}$$

$$\text{and}\quad V_{\mathcal{C}}^{-1}(\mathbf{r}) = \exp\Big(ig\mathcal{X}^\alpha(\mathbf{r})\tfrac{\tau^\alpha}{2}\Big)\,\exp\Big(ig\overline{\mathcal{Y}^\alpha}(\mathbf{r})\tfrac{\tau^\alpha}{2}\Big) \tag{7}$$

$$\text{with}\quad \mathcal{X}^\alpha(\mathbf{r}) = \tfrac{\partial_i}{\partial^2}A_i^\alpha(\mathbf{r}) \quad\text{and}\quad \overline{\mathcal{Y}^\alpha}(\mathbf{r}) = \tfrac{\partial_j}{\partial^2}\overline{\mathcal{A}_j^\alpha}(\mathbf{r}). \tag{8}$$

The composition law for two successive rotations can be used to express $V_{\mathcal{C}}(\mathbf{r})$ in the form

$$V_{\mathcal{C}}(\mathbf{r}) = \exp\Big(-ig\mathcal{Z}^\alpha(\mathbf{r})\tfrac{\tau^\alpha}{2}\Big) \tag{9}$$

where $\mathcal{Z}^\alpha(\mathbf{r})$ is a well-known functional of $\mathcal{X}^\alpha(\mathbf{r})$ and $\overline{\mathcal{Y}^\alpha}(\mathbf{r})$. The transversality of the gauge-invariant field is manifested most directly by transforming Eq. (5) into

$$A^b_{\mathsf{GI}\,i}(\mathbf{r}) = A^b_{T\,i}(\mathbf{r}) + [\delta_{ij} - \tfrac{\partial_i\partial_j}{\partial^2}]\overline{\mathcal{A}_j^b}(\mathbf{r}), \tag{10}$$

demonstrating that $A^b_{\mathsf{GI}\,i}(\mathbf{r})$ is the sum of $A^b_{T\,i}(\mathbf{r})$ — the transverse part of the gauge field $A_i^b(\mathbf{r})$ in the temporal gauge — and the transverse part of the resolvent field. The resolvent field is also required for defining the gauge-invariant quark field

$$\psi_{\mathsf{GI}}(\mathbf{r}) = V_{\mathcal{C}}(\mathbf{r})\,\psi(\mathbf{r}) \quad\text{and}\quad \psi^\dagger_{\mathsf{GI}}(\mathbf{r}) = \psi^\dagger(\mathbf{r})\,V_{\mathcal{C}}^{-1}(\mathbf{r})\,. \tag{11}$$

$V_{\mathcal{C}}(\mathbf{r})$ can be understood as an extension, to non-Abelian gauge theories, of a much simpler but similar operator that made charged QED states gauge-invariant and that was introduced by Dirac.[20] Dirac-like operators implement gauge invariance without introducing path dependence, and it has been argued that such Dirac-like operators have advantages for implementing gauge invariance over other procedures that generate gauge-invariant charged fields with path-dependent line integrals.[21, 22] We have noted that Eqs. (5) and (11) implement gauge invariance — they do not describe gauge transformations.[1] By construction, $V_{\mathcal{C}}(\mathbf{r})$ transforms A_i^a and ψ so that they become invariant to further time-independent gauge-transformations consistent with the temporal gauge condition.[2] Although formally Eqs. (5) and (11) are very suggestive of gauge transformations of gauge and spinor fields respectively by c-number functions, that is not what they are. $V_{\mathcal{C}}(\mathbf{r})$ is itself a functional of gauge fields, and is transformed by the same gauge transformations that transform the fields on which it acts, just as is the case in the corresponding operator Dirac proposed for QED. [20] We will show, in this section, that Eq. (5) describes a gauge field that can be identified with the Coulomb-gauge field; that, in combination with other suitably constructed gauge-invariant quantities, it can be used in the representation of the temporal-gauge Hamiltonian; and that the dynamically effective part of this Hamiltonian can be interpreted as the Coulomb-gauge Hamiltonian.

Eqs. (5)-(11) show that the resolvent field $\overline{\mathcal{A}_j^b}(\mathbf{r})$ has a central role in the representation of QCD in terms of gauge-invariant field variables. $\overline{\mathcal{A}_j^b}(\mathbf{r})$ is determined by a nonlinear integral equation that was obtained in the course of constructing the states that implement the non-Abelian Gauss's law.[1] In subsequent work, this nonlinear integral equation was transformed (subject to an *ansatz*) into a nonlinear differential equation that we solved, resulting in nonperturbative representations of the resolvent field and the gauge-invariant gauge field.[4] We will discuss these nonperturbative solutions in more detail in section 3. An important corollary of the formalism that leads to this equation for the resolvent field is the commutation rule [23]

$$\int d\mathbf{r}\left[\Pi_j^b(\mathbf{y})\,,\overline{\mathcal{A}_k^c}(\mathbf{r})\right]\mathcal{U}_{ki}^{ca}(\mathbf{r}-\mathbf{x})+\mathcal{U}_{ji}^{ba}(\mathbf{y}-\mathbf{x})= \\ \frac{1}{2}\sum_{r=0}\mathsf{Tr}\left[\tau^b V_{\mathcal{C}}^{-1}(\mathbf{y})\tau^s V_{\mathcal{C}}(\mathbf{y})\right](-1)^{r+1}g^r\epsilon_r^{\vec{v}sc}\tfrac{\partial_j}{\partial^2}\left(\mathcal{T}_{(r)\,k}^{\vec{v}}(\mathbf{y})\mathcal{U}_{ki}^{ca}(\mathbf{y}-\mathbf{x})\right) \quad (12)$$

$$\text{where}\quad \mathcal{U}_{ki}^{ca}(\mathbf{y}-\mathbf{x})=\delta_{ca}\mathcal{U}_{ki}(\mathbf{y}-\mathbf{x}) \quad (13)$$

$$\text{with}\quad \mathcal{U}_{ki}(\mathbf{y}-\mathbf{x})=-i\left(\delta_{ik}-\frac{\partial_i\partial_k}{\partial^2}\right)\delta(\mathbf{y}-\mathbf{x}). \quad (14)$$

[2] Gauge transformations within the temporal gauge, for which the Gauss's law operator given in Eq. (1) is the generator, are restricted to time-independent gauge functions, so that the temporal-gauge constraint is not violated.

In Eq. (12), we have used a notation introduced in earlier work: [1, 2, 3] $\epsilon_r^{\vec{v}ba}$ represents the chain of structure constants

$$\epsilon_r^{\vec{v}ba} = \epsilon^{v_1 b s_1}\ \epsilon^{s_1 v_2 s_2}\, \epsilon^{s_2 v_3 s_3}\ \cdots\ \epsilon^{s_{(r-2)} v_{(r-1)} s_{(r-1)}} \epsilon^{s_{(r-1)} v_r a} \tag{15}$$

where repeated superscripted indices are summed from 1→3. For $r = 1$, the chain reduces to $\epsilon_r^{\vec{v}ba} = \epsilon^{vba}$; and for $r = 0$, $\epsilon_r^{\vec{v}ba} = -\delta_{ba}$. Furthermore, we have used

$$\frac{\partial_j}{\partial^2}\left(\mathcal{T}^{\vec{v}}_{(r)\,k}(\mathbf{x})\mathcal{U}_{ki}(\mathbf{x})\right) = \left(\frac{\partial_j}{\partial^2} A^{v_1}_{\mathsf{GI}\,l_1}(\mathbf{x})\,\frac{\partial_{l_1}}{\partial^2}\left(A^{v_2}_{\mathsf{GI}\,l_2}(\mathbf{x})\,\frac{\partial_{l_2}}{\partial^2}\left(\cdots\left(A^{v_{(r-1)}}_{\mathsf{GI}\,l_{(r-1)}}(\mathbf{x})\,\frac{\partial_{l_{(r-1)}}}{\partial^2}\left(\left(A^{v_r}_{\mathsf{GI}\,k}(\mathbf{x})\mathcal{U}_{ki}(\mathbf{x})\right)\right)\right)\right)\right)\right) \tag{16}$$

which, for $r = 1$, reduces to $\frac{\partial_j}{\partial^2}\left(\mathcal{T}^{\vec{v}}_{(1)\,k}(\mathbf{x})\mathcal{U}_{ki}(\mathbf{x})\right) = \frac{\partial_j}{\partial^2}\left(A^{v}_{\mathsf{GI}\,k}(\mathbf{x})\mathcal{U}_{ki}(\mathbf{x})\,\right)$ and for $r = 0$, reduces to $\frac{\partial_j}{\partial^2}\left(\mathcal{T}^{\vec{v}}_{(0)\,k}(\mathbf{x})\mathcal{U}_{ki}(\mathbf{x})\right) = \mathcal{U}_{ji}(\mathbf{x})$. For clarity, we give the explicit form of the r-th element of $\frac{\partial_j}{\partial^2}\left(\mathcal{T}^{\vec{v}}_{(r)\,k}(\mathbf{y})\mathcal{U}_{ki}(\mathbf{y}-\mathbf{x})\right)$, which is:

$$\begin{aligned}\frac{\partial_j}{\partial^2}\left(\mathcal{T}^{\vec{v}}_{(r)\,k}(\mathbf{y})\mathcal{U}_{ki}(\mathbf{y}-\mathbf{x})\right) &= i(-1)^r\frac{\partial}{\partial y_j}\int\frac{d\mathbf{z}(1)}{4\pi|\mathbf{y}-\mathbf{z}(1)|}A^{v_1}_{\mathsf{GI}\,l_1}(\mathbf{z}(1))\frac{\partial}{\partial z_{l_1}(1)}\int\frac{d\mathbf{z}(2)}{4\pi|\mathbf{z}(1)-\mathbf{z}(2)|}\times\\ &A^{v_2}_{\mathsf{GI}\,l_2}(\mathbf{z}(2))\frac{\partial}{\partial z_{l_2}(2)}\ \cdots\int\frac{d\mathbf{z}(r-1)}{4\pi|\mathbf{z}(r-2)-\mathbf{z}(r-1)|}A^{v_{r-1}}_{\mathsf{GI}\,l_{r-1}}(\mathbf{z}(r-1))\frac{\partial}{\partial z_{l_{r-1}}(r-1)}\times\\ &\left(\frac{1}{4\pi|\mathbf{z}(r-1)-\mathbf{x}|}A^{v_r}_{\mathsf{GI}\,i}(\mathbf{x}) + \int d\mathbf{z}\frac{1}{4\pi|\mathbf{z}(r-1)-\mathbf{z}|}A^{v_r}_{\mathsf{GI}\,k}(\mathbf{z})\frac{\partial}{\partial z_k}\frac{\partial}{\partial z_i}\frac{1}{4\pi|\mathbf{z}-\mathbf{x}|}\right);\end{aligned} \tag{17}$$

the leading (0-th) order term is

$$\frac{\partial_j}{\partial^2}\left(\mathcal{T}^{\vec{v}}_{(0)\,k}(\mathbf{y})\mathcal{U}_{ki}(\mathbf{y}-\mathbf{x})\right) = -i\left(\delta_{ij} - \frac{\partial_i\partial_j}{\partial^2}\right)\delta(\mathbf{y}-\mathbf{x})\,. \tag{18}$$

We observe that $\mathcal{U}^{ca}_{ki}(\mathbf{y}-\mathbf{x})$ serves as a projection operator that selects the transverse parts of the fields over which it is integrated, and that Eqs. (10) and (12) in combination show that

$$\int d\mathbf{r}\left[\Pi^b_j(\mathbf{y})\,,\overline{\mathcal{A}^c_k}(\mathbf{r})\right]\mathcal{U}^{ca}_{ki}(\mathbf{r}-\mathbf{x}) + \mathcal{U}^{ba}_{ji}(\mathbf{y}-\mathbf{x}) = \left[\Pi^b_j(\mathbf{y})\,,A^a_{\mathsf{GI}\,i}(\mathbf{x})\right]. \tag{19}$$

The trace $\mathsf{Tr}[\tau^b V_{\mathcal{C}}^{-1}\tau^s V_{\mathcal{C}}]$, which appears on the right-hand-side of Eq. (12), can be expressed as $(\tau^b)_{qn}(V_{\mathcal{C}}^{-1}\tau^s V_{\mathcal{C}})_{nq}$; with the use of the identity

$$\tau^b_{\kappa l}\tau^b_{qn} = 2\delta_{\kappa n}\delta_{lq} - \delta_{\kappa l}\delta_{nq}, \tag{20}$$

we obtain

$$(\tau^b)_{\kappa l}(\tau^b)_{qn}(V_{\mathcal{C}}^{-1}\tau^s V_{\mathcal{C}})_{nq} = 2\left(V_{\mathcal{C}}^{-1}\tau^s V_{\mathcal{C}}\right)_{\kappa l} - \mathsf{Tr}\left[V_{\mathcal{C}}^{-1}\tau^s V_{\mathcal{C}}\right]\delta_{\kappa l}. \tag{21}$$

Since $\mathsf{Tr}[V_{\mathcal{C}}^{-1}\tau^s V_{\mathcal{C}}] = \mathsf{Tr}[\tau^s] = 0$, it follows from Eqs. (12) and (21) that

$$\left[\left(\tau^b\right)_{\kappa l}\Pi^b_j(\mathbf{y})\,,A^a_{\mathsf{GI}\,i}(\mathbf{x})\right] = \sum_{r=0}\left(V_{\mathcal{C}}^{-1}(\mathbf{y})\tau^s V_{\mathcal{C}}(\mathbf{y})\right)_{\kappa l}(-1)^{r+1}g^r\epsilon_r^{\vec{v}sc}\frac{\partial_j}{\partial^2}\left(\mathcal{T}^{\vec{v}}_{(r)\,k}(\mathbf{y})\mathcal{U}^{ca}_{ki}(\mathbf{y}-\mathbf{x})\right) \tag{22}$$

and that, upon contraction with $[V_{\mathcal{C}}(\mathbf{y})\tau^d V_{\mathcal{C}}^{-1}(\mathbf{y})]_{l\kappa}$, we obtain

$$\left[\Pi^d_{\mathsf{GI}\,j}(\mathbf{y})\,,A^a_{\mathsf{GI}\,i}(\mathbf{x})\right]=-i\sum_{r=0}(-1)^{r+1}g^r\epsilon_r^{\vec{v}da}\tfrac{\partial_j}{\partial^2}\left\{\mathcal{T}^{\vec{v}}_{(r)\,k}(\mathbf{y})\left(\delta_{ik}-\frac{\partial_i\partial_k}{\partial^2}\right)\delta(\mathbf{y}-\mathbf{x})\right\} \tag{23}$$

where

$$\Pi^d_{\mathsf{GI}\,i}(\mathbf{y})=\tfrac{1}{2}\mathsf{Tr}[V_{\mathcal{C}}^{-1}(\mathbf{y})\tau^d V_{\mathcal{C}}(\mathbf{y})\tau^b]\Pi^b_i(\mathbf{y}) \tag{24}$$

has been identified in previous work as the gauge-invariant momentum conjugate to the gauge field, (and the negative gauge-invariant chromoelectric field).[3] The following further explanatory remark for Eq. (24) can be given: We let $\Pi_{\mathsf{GI}\,i}=\Pi^a_{\mathsf{GI}\,i}\frac{\tau^a}{2}$, $A_{\mathsf{GI}\,i}=A^a_{\mathsf{GI}\,i}\frac{\tau^a}{2}$, $A_i=A^a_i\frac{\tau^a}{2}$, and make use of the analogy of Eq. (5) to a gauge transformation to define a corresponding $A_{\mathsf{GI}\,0}$, and to observe that, since in the temporal gauge $A_0=0$,

$$A_{\mathsf{GI}\,0}=-\frac{i}{g}\mathcal{W}_0 \quad \text{where} \quad \mathcal{W}_0=V_{\mathcal{C}}\partial_0 V_{\mathcal{C}}^{-1}\,.$$

We recall an operator-order we have previously found necessary, and impose it on the obvious definition of a "gauge-invariant momentum (and negative chromoelectric field)"

$$\Pi_{\mathsf{GI}\,i}=||\partial_i A_{\mathsf{GI}\,0}+\partial_0 A_{\mathsf{GI}\,\mathsf{i}}-ig\,[A_{\mathsf{GI}\,\mathsf{i}},A_{\mathsf{GI}\,0}]\,||\,, \tag{25}$$

where, using a notation introduced in Ref.[1], bracketing between double bars denotes a normal order in which all gauge fields and functionals of gauge fields appear to the left of all momenta conjugate to gauge fields, but where that order is imposed only after all indicated commutators have been evaluated (including the commutator implied by the derivatives ∂_0 and ∂_i). We observe that

$$||\partial_0 A_{\mathsf{GI}\,\mathsf{i}}||=||\left[V_{\mathcal{C}}A_iV_{\mathcal{C}}^{-1},\mathcal{W}_0\right]+V_{\mathcal{C}}\partial_0A_iV_{\mathcal{C}}^{-1}+\frac{i}{g}\left(\partial_0V_{\mathcal{C}}\partial_iV_{\mathcal{C}}^{-1}+V_{\mathcal{C}}\partial_0\partial_iV_{\mathcal{C}}^{-1}\right)||\,, \tag{26}$$

$$||\partial_i A_{\mathsf{GI}\,0}||=-||\frac{i}{g}\left(\partial_iV_{\mathcal{C}}\partial_0V_{\mathcal{C}}^{-1}+V_{\mathcal{C}}\partial_0\partial_iV_{\mathcal{C}}^{-1}\right)||\,, \tag{27}$$

$$\text{and}\qquad -ig\,[A_{\mathsf{GI}\,\mathsf{i}},A_{\mathsf{GI}\,0}]=-\left[V_{\mathcal{C}}A_iV_{\mathcal{C}}^{-1},\mathcal{W}_0\right]-\frac{i}{g}\left[\partial_0V_{\mathcal{C}},\partial_iV_{\mathcal{C}}^{-1}\right]\,. \tag{28}$$

Combining Eqs. (26-28), we find that

$$\Pi_{\mathsf{GI}\,i}=||V_{\mathcal{C}}\partial_0A_iV_{\mathcal{C}}^{-1}||=||V_{\mathcal{C}}\Pi_iV_{\mathcal{C}}^{-1}||=V_{\mathcal{C}}\tfrac{\tau^b}{2}V_{\mathcal{C}}^{-1}\Pi^b_i\,, \tag{29}$$

which agrees with Eq. (24). This demonstration is not necessary to prove that $\Pi^a_{\mathsf{GI}\,i}$ is gauge-invariant — the transformation properties of Π^a_i and of $V_{\mathcal{C}}\frac{\tau^b}{2}V_{\mathcal{C}}^{-1}$ suffice for that — but it nevertheless makes the identification of $\Pi^a_{\mathsf{GI}\,i}$ as the gauge-invariant momentum conjugate to $A^a_{\mathsf{GI}\,i}$ more understandable.

As we will point out later in this section, the commutator given in Eq. (23) agrees with the corresponding commutator for the gauge field and its conjugate momentum in the Coulomb gauge, given by Schwinger. [5] The gauge field in the Coulomb gauge and the gauge-invariant gauge field $A^b_{\mathsf{GI}\,i}(\mathbf{r})$ constructed within the temporal gauge are both transverse.[3] In Ref.[5] and in our work, the momentum conjugate to the gauge field — the negative chromoelectric field — has a longitudinal component, which is needed to implement Gauss's law. The Coulomb-gauge commutation rules in Refs. [7, 8, 9] differ from Eq. (23), because in these works, the momentum conjugate to the Coulomb-gauge field has been defined to be transverse.

We next turn to a discussion of the "gauge-invariant Gauss's law operator" — the Gauss's law operator in which gauge-invariant fields replace the original gauge-dependent ones — which we define as

$$\hat{\mathcal{G}}^d_{\mathsf{GI}}(\mathbf{r}) = \partial_i \Pi^d_{\mathsf{GI}\,i}(\mathbf{r}) + g\epsilon^{duv} A^u_{\mathsf{GI}\,i}(\mathbf{r}) \Pi^v_{\mathsf{GI}\,i}(\mathbf{r}) + j^d_{0\ \mathsf{GI}}(\mathbf{r})\,. \tag{30}$$

In particular, we want to investigate whether the use of the gauge-invariant Gauss's law operator as the generator of gauge transformations is consistent with the gauge invariance of $A^b_{\mathsf{GI}\,i}(\mathbf{r})$. We will demonstrate this consistency in this section.

We will first demonstrate a simple relation between $\hat{\mathcal{G}}^d_{\mathsf{GI}}(\mathbf{r})$ and $\hat{\mathcal{G}}^d(\mathbf{r})$, from which

$$\int d\mathbf{r} \left[\hat{\mathcal{G}}^b_{\mathsf{GI}}(\mathbf{r})\,,A^a_{\mathsf{GI}\,i}(\mathbf{x})\right]\,\delta\omega^b(\mathbf{r}) = 0 \tag{31}$$

is an immediate consequence. We observe that

$$\begin{aligned}\partial_i \Pi^d_{\mathsf{GI}\,i}(\mathbf{r}) = {} & \tfrac{1}{2}\mathsf{Tr}\left\{\tau^d \partial_i V_{\mathcal{C}}(\mathbf{r}) \tau^b V^{-1}_{\mathcal{C}}(\mathbf{r})\right\} \Pi^b_i(\mathbf{r}) + \tfrac{1}{2}\mathsf{Tr}\left\{\tau^d V_{\mathcal{C}}(\mathbf{r}) \tau^b \partial_i V^{-1}_{\mathcal{C}}(\mathbf{r})\right\} \Pi^b_i(\mathbf{r}) + \\ & \tfrac{1}{2}\mathsf{Tr}\left\{\tau^d V_{\mathcal{C}}(\mathbf{r}) \tau^b V^{-1}_{\mathcal{C}}(\mathbf{r})\right\} \partial_i \Pi^b_i(\mathbf{r})\end{aligned} \tag{32}$$

and that, for $\chi_i(\mathbf{r}) = V_{\mathcal{C}}(\mathbf{r})\partial_i V^{-1}_{\mathcal{C}}(\mathbf{r})$,

$$\partial_i \Pi^d_{\mathsf{GI}\,i} = \tfrac{1}{2}\mathsf{Tr}\left\{\tau^d \left[V_{\mathcal{C}} \tau^b V^{-1}_{\mathcal{C}}\,,\chi_i\right]\right\} \Pi^b_i + \tfrac{1}{2}\mathsf{Tr}\left\{\tau^d V_{\mathcal{C}} \tau^b V^{-1}_{\mathcal{C}}\right\} \partial_i \Pi^b_i. \tag{33}$$

In an Appendix, we will show that we can set

$$\chi_i = \tfrac{i}{2}\tau^u \mathcal{P}_{ui} \quad \text{and} \quad V_{\mathcal{C}} \tau^b V^{-1}_{\mathcal{C}} = \tau^v \mathcal{R}_{vb}, \tag{34}$$

where $\mathcal{P}_{ui}$ and $\mathcal{R}_{vb}$ are functions of gauge fields only — they are independent of the canonical momentum Π^a_i and also contain no further SU(2) generators — so that

$$\begin{aligned}\tfrac{1}{2}\mathsf{Tr}\left\{\tau^d \left[V_{\mathcal{C}} \tau^b V^{-1}_{\mathcal{C}}\,,\chi_i\right]\right\} \Pi^b_i \quad & = \tfrac{i}{4}\mathsf{Tr}\left\{\tau^d \left[\tau^v\,,\tau^u\right]\right\} \mathcal{R}_{vb} \mathcal{P}_{ui} \Pi^b_i \\ & = \epsilon^{duv} \mathcal{R}_{vb} \mathcal{P}_{ui} \Pi^b_i\end{aligned} \tag{35}$$

[3]The relation between gauge-invariance and transversality can be inverted to produce a perturbative representation of the gauge-invariant field, as in Refs.[21, 24].

and

$$\partial_i \Pi^d_{\mathsf{GI}\,i} = \epsilon^{duv}\mathcal{R}_{vb}\mathcal{P}_{ui}\Pi^b_i + \mathcal{R}_{db}\partial_i\Pi^b_i. \tag{36}$$

Similarly, using Eqs. (5), (24), and (34), we can express $g\epsilon^{duv}A^u_{\mathsf{GI}\,i}\Pi^v_{\mathsf{GI}\,i}$ as

$$g\epsilon^{duv}A^u_{\mathsf{GI}\,i}\Pi^v_{\mathsf{GI}\,i} = g\epsilon^{duv}\left(\mathcal{R}_{ua}\mathcal{R}_{vb}A^a_i - \tfrac{1}{g}\mathcal{R}_{vb}\mathcal{P}_{ui}\right)\Pi^b_i \tag{37}$$

and find that $\epsilon^{duv}\mathcal{R}_{vb}\mathcal{P}_{ui}\Pi^b_i$ terms in Eqs. (36) and (37) cancel, so that the gauge-invariant Gauss's law operator can be expressed as

$$\hat{\mathcal{G}}^d_{\mathsf{GI}}(\mathbf{r}) = \mathcal{R}_{db}\partial_i\Pi^b_i + g\epsilon^{duv}\mathcal{R}_{ua}\mathcal{R}_{vb}A^a_i\Pi^b_i + \mathcal{R}_{db}j^b_0, \tag{38}$$

where, in order to obtain the last term in Eq. (38), we have used[4]

$$j^d_{0\,\mathsf{GI}} = g\psi^\dagger V_{\mathcal{C}}^{-1}\tau^d V_{\mathcal{C}}\psi = \mathcal{R}_{db}g\psi^\dagger\tau^b\psi = \mathcal{R}_{db}j^b_0 \tag{39}$$

which we justify in the Appendix. Eq. (34) leads to

$$\mathcal{R}_{db} = \tfrac{1}{2}\mathsf{Tr}[\tau^d V_{\mathcal{C}}\tau^b V_{\mathcal{C}}^{-1}], \tag{40}$$

so that

$$\epsilon^{duv}\mathcal{R}_{ua}\mathcal{R}_{vb} = \tfrac{1}{4}\epsilon^{duv}(\tau^u)_{ij}(\tau^v)_{kp}\left(V_{\mathcal{C}}\tau^a V_{\mathcal{C}}^{-1}\right)_{ji}\left(V_{\mathcal{C}}\tau^b V_{\mathcal{C}}^{-1}\right)_{pk}. \tag{41}$$

We make use of the identity

$$\epsilon^{duv}(\tau^u)_{ij}(\tau^v)_{kp} = i\left(\delta_{jk}(\tau^d)_{ip} - \delta_{ip}(\tau^d)_{kj}\right) \tag{42}$$

to obtain

$$\epsilon^{duv}\mathcal{R}_{ua}\mathcal{R}_{vb} = \tfrac{i}{4}\mathsf{Tr}\left\{\tau^d V_{\mathcal{C}}\left[\tau^b\,,\tau^a\right]V_{\mathcal{C}}^{-1}\right\} = \epsilon^{baq}\mathcal{R}_{dq}, \tag{43}$$

and after relabeling dummy indices, observe that Eqs. (38) and (43) lead to

$$\hat{\mathcal{G}}^d_{\mathsf{GI}} = \mathcal{R}_{db}\left(\partial_i\Pi^b_i + g\epsilon^{buv}A^u_i\Pi^v_i + j^b_0\right) = \mathcal{R}_{db}\hat{\mathcal{G}}^b. \tag{44}$$

Since $\mathcal{R}_{ab}$ trivially commutes with $A^b_{\mathsf{GI}\,i}$, the fact that $\hat{\mathcal{G}}^d$ commutes with $A^b_{\mathsf{GI}\,i}$ is sufficient for the demonstration that $\hat{\mathcal{G}}^d_{\mathsf{GI}}$ also commutes with $A^b_{\mathsf{GI}\,i}$, thus proving Eq. (31).

Another, very direct demonstration of Eq. (31) begins with the use of Eq. (23) to obtain

$$\int d\mathbf{r}\left[\partial_j\Pi^b_{\mathsf{GI}\,j}(\mathbf{r})\,,A^a_{\mathsf{GI}\,i}(\mathbf{x})\right]\delta\omega^b(\mathbf{r}) = \int d\mathbf{r}\sum_{r=0}(-1)^{r+1}g^r\epsilon^{\vec{v}ba}_r\left(\mathcal{T}^{\vec{v}}_{(r)\,k}(\mathbf{r})\mathcal{U}_{ki}(\mathbf{r}-\mathbf{x})\right)\delta\omega^b(\mathbf{r})\,; \tag{45}$$

[4] To simplify this argument, we make use in this discussion of a representation defined in Ref.[1] — the so-called $\mathcal{C}$-representation — in which $V_{\mathcal{C}}(\mathbf{r})\psi(\mathbf{r})$ is the gauge-invariant spinor and in which $j^a_{0\,\mathsf{GI}} = g\psi^\dagger V_{\mathcal{C}}^{-1}(\tau^a/2)V_{\mathcal{C}}\psi$ is the gauge-invariant color charge density. Elsewhere in this paper, we have used the so-called $\mathcal{N}$-representation, in which the $V_{\mathcal{C}}$ transformation has already been implicitly carried out for the quark field (but *not* the gauge field), so that it is the quark field ψ and $j^a_0 = g\psi^\dagger(\tau^a/2)\psi$ that are gauge invariant.

we observe that the $r=0$ term on the right-hand-side of Eq. (45) vanishes, because the degenerate values listed immediately preceding and following Eq. (16) demonstrate that $\partial_j(\frac{\partial_j}{\partial^2}\mathcal{T}^{\vec{v}}_{(0)\,k}(\mathbf{r})\mathcal{U}_{ki})=0$. We can therefore begin the sum with $r=1$ instead of with $r=0$, redefine the dummy index r to be $r+1$, and then initiate the sum with $r=0$ for the new index r, obtaining

$$\int d\mathbf{y}\left[\partial_j\Pi^b_{\mathsf{GI}\,j}(\mathbf{y})\,,A^a_{\mathsf{GI}\,i}(\mathbf{x})\right]\delta\omega^b(\mathbf{y})=\int d\mathbf{y}\sum_{r=0}(-1)^r g^{r+1}\epsilon^{\vec{v}ba}_{r+1}\left(\mathcal{T}^{\vec{v}}_{r+1\,k}(\mathbf{y})\mathcal{U}_{ki}(\mathbf{y}-\mathbf{x})\right)\delta\omega^b(\mathbf{y}). \tag{46}$$

We can also evaluate

$$g\int d\mathbf{y}\epsilon^{bcu}A^c_{\mathsf{GI}\,j}(\mathbf{y})\left[\Pi^u_{\mathsf{GI}\,j}(\mathbf{y})\,,A^a_{\mathsf{GI}\,i}(\mathbf{x})\right]\delta\omega^b(\mathbf{y})=\int d\mathbf{y}\sum_{r=0}(-1)^{r+1}g^{r+1}\epsilon^{bcu}\epsilon^{\vec{v}ua}_r\times$$
$$A^c_{\mathsf{GI}\,j}(\mathbf{y})\frac{\partial_j}{\partial^2}\left(\mathcal{T}^{\vec{v}}_{(r)\,k}(\mathbf{y})\mathcal{U}_{ki}(\mathbf{y}-\mathbf{x})\right)\delta\omega^b(\mathbf{y}) \tag{47}$$

and observe that

$$\epsilon^{bcu}\epsilon^{\vec{v}ua}_r A^c_{\mathsf{GI}\,j}(\mathbf{y})\frac{\partial_j}{\partial^2}\left(\mathcal{T}^{\vec{v}}_{(r)\,k}(\mathbf{y})\mathcal{U}_{ki}(\mathbf{y}-\mathbf{x})\right)=\epsilon^{\vec{v}ba}_{r+1}\left(\mathcal{T}^{\vec{v}}_{(r+1)\,k}(\mathbf{y})\mathcal{U}_{ki}(\mathbf{y}-\mathbf{x})\right) \tag{48}$$

so that

$$g\int d\mathbf{y}\epsilon^{bcu}A^c_{\mathsf{GI}\,j}(\mathbf{y})\left[\Pi^u_{\mathsf{GI}\,j}(\mathbf{y})\,A^a_{\mathsf{GI}\,i}(\mathbf{x})\right]\delta\omega^b(\mathbf{y})=\int d\mathbf{y}\sum_{r=0}(-1)^{r+1}g^{r+1}\epsilon^{\vec{v}ba}_{r+1}\left(\mathcal{T}^{\vec{v}}_{(r+1)\,k}(\mathbf{y})\mathcal{U}_{ki}(\mathbf{y}-\mathbf{x})\right)\delta\omega^b(\mathbf{y}). \tag{49}$$

Cancellation between the right-hand sides of Eqs. (46) and (49) verifies Eq. (31), and therefore confirms that the use of $\hat{\mathcal{G}}^d_{\mathsf{GI}}$ as the generator of infinitesimal gauge transformations is consistent with the gauge invariance of $A^a_{\mathsf{GI}\,i}(\mathbf{x})$ (but not with the gauge invariance of $\Pi^a_{\mathsf{GI}\,i}(\mathbf{y})$).

The mathematical apparatus we developed for constructing gauge-invariant fields enables us to express the QCD Hamiltonian entirely in terms of these gauge-invariant fields and $\Pi^b_{\mathsf{GI}\,j}$ (the gauge-invariant negative chromoelectric field). [2, 3] The QCD Hamiltonian, represented in this way, has the form

$$\hat{H}_{\mathsf{GI}}=H_{\mathsf{GI}}+H_{\mathcal{G}} \tag{50}$$

where $H_{\mathcal{G}}$ annihilates states that implement Gauss's law and has no dynamical consequences in QCD. H_{GI} is the effective Hamiltonian in this representation of QCD and is given by

$$H_{\mathsf{GI}}=\int d\mathbf{r}\left[\,\tfrac{1}{2}\Pi^{a\,\dagger}_{\mathsf{GI}\,i}(\mathbf{r})\Pi^a_{\mathsf{GI}\,i}(\mathbf{r})+\tfrac{1}{4}F^a_{\mathsf{GI}\,ij}(\mathbf{r})F^a_{\mathsf{GI}\,ij}(\mathbf{r})+\psi^\dagger(\mathbf{r})\left(\beta m-i\alpha_i\partial_i\right)\psi(\mathbf{r})\right]+\tilde{H}'\,, \tag{51}$$

where, in this case, ψ and $\psi^\dagger$ denote the gauge-invariant spinor (quark) fields, and where

$$F^a_{\mathsf{GI}\,ij}(\mathbf{r})=\partial_jA^a_{\mathsf{GI}\,i}(\mathbf{r})-\partial_iA^a_{\mathsf{GI}\,j}(\mathbf{r})-g\epsilon^{abc}A^b_{\mathsf{GI}\,i}(\mathbf{r})A^c_{\mathsf{GI}\,j}(\mathbf{r}). \tag{52}$$

$\tilde{H}'$ is given by

$$\tilde{H}' = \int d\mathbf{r} \left(\tfrac{1}{2} J^{a\,\dagger}_{0\,(\mathsf{GI})}(\mathbf{r}) \frac{1}{\partial^2} \mathcal{K}^a_0(\mathbf{r}) + \tfrac{1}{2}\, \mathcal{K}^a_0(\mathbf{r}) \frac{1}{\partial^2} J^a_{0\,(\mathsf{GI})} - \tfrac{1}{2} \mathcal{K}^a_0(\mathbf{r}) \frac{1}{\partial^2} \mathcal{K}^a_0(\mathbf{r}) - j^a_i(\mathbf{r}) A^a_{\mathsf{GI}\,i}(\mathbf{r}) \right) \tag{53}$$

$$\text{where} \quad J^a_{0\,(\mathsf{GI})} = g\epsilon^{abc} A^b_{\mathsf{GI}\,i}(\mathbf{r}) \Pi^c_{\mathsf{GI}\,i}(\mathbf{r}) \tag{54}$$

is the gauge-invariant glue color-charge density, $j^a_i(\mathbf{r}) = g\psi^\dagger(\mathbf{r})\alpha_i \frac{\tau^a}{2}\psi(\mathbf{r})$ is the gauge-invariant quark color-current density in this representation, and where

$$\mathcal{K}^d_0(\mathbf{r}) = \sum_{n=0} \epsilon^{\vec{\delta}dh}_{(n)} (-1)^n g^n \left(\mathcal{T}^{\vec{\delta}}_{(n)}(\mathbf{r}) j^h_0(\mathbf{r}) \right) , \tag{55}$$

$$\text{with} \;\; \mathcal{T}^{\vec{\delta}}_{(n)}(\mathbf{r}) j^a_0(\mathbf{r}) = A^{\delta(1)}_{\mathsf{GI}\,j(1)}(\mathbf{r}) \, \tfrac{\partial_{j(1)}}{\partial^2} \left(A^{\delta(2)}_{\mathsf{GI}\,j(2)}(\mathbf{r}) \, \tfrac{\partial_{j(2)}}{\partial^2} \left(\cdots \left(A^{\delta(n)}_{\mathsf{GI}\,j(n)}(\mathbf{r}) \, \tfrac{\partial_{j(n)}}{\partial^2} \left(j^a_0(\mathbf{r}) \right) \right) \right) \right) . \tag{56}$$

The more transparent explicit form of $\mathcal{K}^d_0(\mathbf{r})$ is:

$$\begin{aligned}
\mathcal{K}^b_0(\mathbf{r}) \quad & = -j^b_0(\mathbf{r}) + g\epsilon^{v(1)ba} A^{v(1)}_{\mathsf{GI}\,i}(\mathbf{r}) \frac{\partial}{\partial r_i} \int \frac{d\mathbf{x}}{4\pi|\mathbf{r}-\mathbf{x}|}\, j^a_0(\mathbf{x}) + \\
& \quad g^2 \epsilon^{v(1)bs(1)} \epsilon^{s(1)v(2)a} A^{v(1)}_{\mathsf{GI}\,i}(\mathbf{r}) \frac{\partial}{\partial r_i} \int \frac{d\mathbf{y}}{4\pi|\mathbf{r}-\mathbf{y}|}\, A^{v(2)}_{\mathsf{GI}\,j}(\mathbf{y}) \frac{\partial}{\partial y_j} \int \frac{d\mathbf{x}}{4\pi|\mathbf{y}-\mathbf{x}|}\, j^a_0(\mathbf{x}) + \cdots \\
+ \quad & \quad g^n \epsilon^{v(1)bs(1)} \cdots \epsilon^{s(n-2)v(n-1)s(n-1)} \epsilon^{s(n-1)v(n)a} A^{v(1)}_{\mathsf{GI}\,i}(\mathbf{r}) \frac{\partial}{\partial r_i} \int \frac{d\mathbf{y}_{(1)}}{4\pi|\mathbf{r}-\mathbf{y}_{(\mathbf{1})}|} \cdots \times \\
& \quad A^{v(n-2)}_{\mathsf{GI}\,\ell}(\mathbf{y}_{(n-3)}) \frac{\partial}{\partial y_{(n-3)\,\ell}} \int \frac{d\mathbf{y}_{(n-2)}}{4\pi|\mathbf{y}_{(n-3)}-\mathbf{y}_{(n-2)}|}\, A^{v(n-1)}_{\mathsf{GI}\,j}(\mathbf{y}_{(n-2)}) \frac{\partial}{\partial y_{(n-2)\,j}} \times \\
& \quad \int \frac{d\mathbf{y}_{(n-1)}}{4\pi|\mathbf{y}_{(n-2)}-\mathbf{y}_{(n-1)}|} A^{v(n)}_{\mathsf{GI}\,k}(\mathbf{y}_{(n-1)}) \frac{\partial}{\partial y_{(n-1)\,k}} \int \frac{d\mathbf{x}}{4\pi|\mathbf{y}_{(n-1)}-\mathbf{x}|}\, j^a_0(\mathbf{x}) + \cdots \, .
\end{aligned} \tag{57}$$

In formulating Eqs. (51)-(57), we note that while it is trivial that $A^a_{\mathsf{GI}i}(\mathbf{r})$ is hermitian, and while we will show that $\hat{\mathcal{G}}^b_{\mathsf{GI}}(\mathbf{r})$ is hermitian as well, $\Pi^{b\,\dagger}_{\mathsf{GI}\,i}(\mathbf{r})$ and $\Pi^b_{\mathsf{GI}\,i}(\mathbf{r})$ are not identical. [5] As can be seen from Eq. (24), in order for $\Pi^{b\,\dagger}_{\mathsf{GI}\,i}(\mathbf{r})$ to be hermitian, $\Pi^b_i(\mathbf{r})$ would have to commute with $R_{db}(\mathbf{r})$ (defined in Eq. (40)). We therefore distinguish between $\Pi^{b\,\dagger}_{\mathsf{GI}\,i}(\mathbf{r})$ and $\Pi^b_{\mathsf{GI}\,i}(\mathbf{r})$ in this discussion; similarly, Eq. (23) shows that J^a_0 and $J^{a\,\dagger}_0$ will differ. The Hamiltonian and its components, $\hat{H}_{\mathsf{GI}}$, H_{GI}, $H_{\mathcal{G}}$, and $\tilde{H}'$, are all manifestly hermitian.

Eq. (53) is very suggestive of an interaction Hamiltonian in the Coulomb gauge, but with $\mathcal{K}^a_0$ appearing in the position in which one might expect to find the charge density of the matter field j^a_0. $\mathcal{K}^a_0$ contains j^a_0 as a crucial component, but extends the latter's dynamical effect over a greater region in space than j^a_0 itself occupies, through a series of

[5] a similar point is made in Ref.[9]; note, however, that the chromoelectric field in Ref.[9] is transverse, while our chromoelectric field includes its longitudinal part, as in Ref.[5]; the longitudinal part of the chromoelectric field accounts for its lack of hermiticity.

"chains" of interactions in which each link has the form $\epsilon^{\alpha v\beta}A^v_{\mathsf{GI}\,i}\frac{\partial_i}{\partial^2}$. The effect of these chains of interactions is that if, for example, j_0^a describes quark color-charges that are limited to a relatively small volume, $\mathcal{K}_0^a$ could be significant over a substantially larger region of space. Further investigation of these nonlocal interactions can be facilitated by the observation that

$$\mathcal{K}_0^a + g\epsilon^{avb}A_{\mathsf{GI}\,i}\frac{\partial_i}{\partial^2}\mathcal{K}_0^b = -j_0^a. \tag{58}$$

The issue here is not that $\mathcal{K}_0^a$ is gauge-invariant. In fact, in the representation used to express $\tilde{H}'$, both $\mathcal{K}_0^a$ and j_0^a are gauge-invariant, as has been discussed in Ref. [3].

A further significant feature of $\tilde{H}'$ is that, besides the nonlocal interaction $-\frac{1}{2}\mathcal{K}_0^a\frac{1}{\partial^2}\mathcal{K}_0^a$, it also includes additional interactions of the gauge-invariant glue color-charge density J_0^a with $\mathcal{K}_0^a$, so that quark and glue color charge-densities interact with each other in the non-Abelian theory in which the gauge field, as well as the matter fields to which it couples, carry color-charge.

$H_{\mathcal{G}}$ — the part of the Hamiltonian that annihilates states that implement Gauss's law and therefore has no dynamical consequences — can be given as

$$H_{\mathcal{G}} = -\tfrac{1}{2}\int d\mathbf{r}\left[\mathcal{G}^a_{\mathsf{GI}}\frac{1}{\partial^2}\mathcal{K}_0^a(\mathbf{r}) + \mathcal{K}_0^a(\mathbf{r})\frac{1}{\partial^2}\mathcal{G}^a_{\mathsf{GI}}\right]. \tag{59}$$

Since $\mathcal{G}^a_{\mathsf{GI}}$ is hermitian, and since any state $|\Psi\rangle$ for which $\mathcal{G}^a_{\mathsf{GI}}(\mathbf{x})\,|\Psi\rangle = 0$ will time-evolve so that $\mathcal{G}^a_{\mathsf{GI}}(\mathbf{x})\,\exp(-i\hat{H}_{\mathsf{GI}}\,t)|\Psi\rangle = 0$ as well — properties of $\mathcal{G}^a_{\mathsf{GI}}$ that we will prove later in this section — $H_{\mathcal{G}}$ will not contribute to matrix elements for physical processes or have any affect on the properties of states in the space of "physical" states which must implement the non-Abelian Gauss's law.

To demonstrate the hermiticity of $\mathcal{G}^b_{\mathsf{GI}}(\mathbf{r})$, we use Eq. (44) to show that

$$\Delta_g(\mathbf{r}) = \mathcal{G}^{d\,\dagger}_{\mathsf{GI}}(\mathbf{r}) - \mathcal{G}^d_{\mathsf{GI}}(\mathbf{r}) = \left[\mathcal{G}^b(\mathbf{r})\,, R_{db}(\mathbf{r})\right]. \tag{60}$$

From the proof that $V_{\mathcal{C}}(\mathbf{r})$ transforms gauge-dependent temporal-gauge quark and gauge fields into gauge-invariant fields as shown in Eqs. (5) and (11),[25] we observe that

$$[\mathcal{G}^c(\mathbf{y})\,, V_{\mathcal{C}}(\mathbf{x})] = gV_{\mathcal{C}}(\mathbf{x})\tfrac{\tau^c}{2}\delta(\mathbf{x}-\mathbf{y}) \quad \text{and} \quad \left[\mathcal{G}^c(\mathbf{y})\,, V_{\mathcal{C}}^{-1}(\mathbf{x})\right] = -g\tfrac{\tau^c}{2}V_{\mathcal{C}}^{-1}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\,, \tag{61}$$

so that

$$\Delta_g(\mathbf{r}) = \tfrac{1}{2}\mathsf{Tr}\left\{\tau^b\left[\mathcal{G}^c(\mathbf{r})\,, V_{\mathcal{C}}(\mathbf{r})\tau^c V_{\mathcal{C}}^{-1}(\mathbf{r})\right]\right\} = 0 \tag{62}$$

and $\mathcal{G}^d_{\mathsf{GI}}(\mathbf{r})$ is shown to be hermitian.[6]

[6] We interpret the commutators $[\mathcal{G}^c(\mathbf{r})\,, V_{\mathcal{C}}(\mathbf{r})]$ and $[\mathcal{G}^c(\mathbf{r})\,, V_{\mathcal{C}}^{-1}(\mathbf{r})]$ as $\lim_{\mathbf{r}'\to\mathbf{r}}[\mathcal{G}^c(\mathbf{r}')\,, V_{\mathcal{C}}(\mathbf{r})]$ and $\lim_{\mathbf{r}'\to\mathbf{r}}[\mathcal{G}^c(\mathbf{r}')\,, V_{\mathcal{C}}^{-1}(\mathbf{r})]$ respectively to regularize the 0 argument of the delta-function.

We will examine the hermiticity of $\Pi^a_{\mathsf{GI}\,i}(\mathbf{x})$ by first evaluating $[\Pi^a_{\mathsf{GI}\,i}(\mathbf{x})\,,\Pi^b_{\mathsf{GI}\,i}(\mathbf{y})]$ and comparing it with the commutator $[\Pi^a_{\mathsf{GI}\,i}(\mathbf{x})\,,\Pi^{b\,\dagger}_{\mathsf{GI}\,i}(\mathbf{y})]$. In this way, we avoid the necessity of evaluating the singular commutator $[\Pi^a_i(\mathbf{x})\,,R_{ba}(\mathbf{x})]$. To evaluate these commutators, we first replace $j^a_0(\mathbf{r})$ by $-g\frac{\tau^a}{2}\delta(\mathbf{r}-\mathbf{x})$ in Eqs. (7) and (19) in Ref.[2], a step justified by the fact that both sets of quantities obey the same closed commutator algebra. In this way, we obtain

$$V_{\mathcal{C}}(\mathbf{x})\Pi^b_j(\mathbf{y})V^{-1}_{\mathcal{C}}(\mathbf{x}) = \Pi^b_j(\mathbf{y}) + \sum_{n=0} g^n(-1)^n R_{db}(\mathbf{y})\epsilon^{\vec{\delta}dh}_n\tfrac{\partial_j}{\partial^2}\left(\mathcal{T}^{\vec{\delta}}_{(n)}(\mathbf{y})g\tfrac{\tau^h}{2}\delta(\mathbf{y}-\mathbf{x})\right) \qquad (63)$$

and therefore that

$$\left[\Pi^b_j(\mathbf{y})\,,V_{\mathcal{C}}(\mathbf{x})\tau^a V^{-1}_{\mathcal{C}}(\mathbf{x})\right] = \sum_{n=0} g^{n+1}(-1)^{n+1}R_{db}(\mathbf{y})\epsilon^{\vec{\delta}dh}_n\tfrac{\partial_j}{\partial^2}\left\{\mathcal{T}^{\vec{\delta}}_{(n)}(\mathbf{y})\delta(\mathbf{y}-\mathbf{x})\left[\tfrac{\tau^h}{2}\,,V_{\mathcal{C}}(\mathbf{x})\tau^a V^{-1}_{\mathcal{C}}(\mathbf{x})\right]\right\} \qquad (64)$$

from which

$$\left[\Pi^b_j(\mathbf{y})\,,R_{\alpha a}(\mathbf{x})\right] = i\sum_{n=0} g^{n+1}(-1)^{n+1}R_{db}(\mathbf{y})\epsilon^{\vec{\delta}dh}_n\epsilon^{hc\alpha}\tfrac{\partial_j}{\partial^2}\left\{\mathcal{T}^{\vec{\delta}}_{(n)}(\mathbf{y})\delta(\mathbf{y}-\mathbf{x})R_{ca}(\mathbf{x})\right\} \qquad (65)$$

follows; and, using $R_{cb}(\mathbf{y})R_{db}(\mathbf{y}) = \delta_{cd}$, we also obtain

$$\left[\Pi^\beta_{\mathsf{GI}\,j}(\mathbf{y})\,,R_{\alpha a}(\mathbf{x})\right] = i\sum_{n=0} g^{n+1}(-1)^{n+1}\epsilon^{\vec{\delta}\beta h}_n\epsilon^{hc\alpha}\tfrac{\partial_j}{\partial^2}\left\{\mathcal{T}^{\vec{\delta}}_{(n)}(\mathbf{y})\delta(\mathbf{y}-\mathbf{x})R_{ca}(\mathbf{x})\right\}\,. \qquad (66)$$

Since

$$\left[\Pi^\alpha_{\mathsf{GI}\,i}(\mathbf{x})\,,\Pi^\beta_{\mathsf{GI}\,j}(\mathbf{y})\right] = \left[\Pi^\alpha_{\mathsf{GI}\,i}(\mathbf{x})\,,R_{\beta b}(\mathbf{y})\right]\Pi^b_j(\mathbf{y}) - \left[\Pi^\beta_{\mathsf{GI}\,j}(\mathbf{y})\,,R_{\alpha a}(\mathbf{x})\right]\Pi^a_i(\mathbf{x})\,, \qquad (67)$$

it follows that

$$\left[\Pi^\alpha_{\mathsf{GI}\,i}(\mathbf{x})\,,\Pi^\beta_{\mathsf{GI}\,j}(\mathbf{y})\right] = i\sum_{n=0} g^{n+1}(-1)^{n+1}\Big[\epsilon^{\vec{\delta}\alpha h}_n\epsilon^{h\gamma\beta}\tfrac{\partial_i}{\partial^2}\left\{\mathcal{T}^{\vec{\delta}}_{(n)}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\Pi^\gamma_{\mathsf{GI}\,j}(\mathbf{y})\right\}$$
$$- \epsilon^{\vec{\delta}\beta h}_n\epsilon^{h\gamma\alpha}\tfrac{\partial_j}{\partial^2}\left\{\mathcal{T}^{\vec{\delta}}_{(n)}(\mathbf{y})\delta(\mathbf{y}-\mathbf{x})\Pi^\gamma_{\mathsf{GI}\,i}(\mathbf{x})\right\}\Big]\,. \qquad (68)$$

The explicit form of the n-th order term $\frac{\partial_j}{\partial^2}\left\{\mathcal{T}^{\vec{\delta}}_{(n)}(\mathbf{y})\delta(\mathbf{y}-\mathbf{x})\Pi^\gamma_{\mathsf{GI}\,i}(\mathbf{x})\right\}$ is

$$\frac{\partial_j}{\partial^2}\left\{\mathcal{T}^{\vec{\delta}}_{(n)}(\mathbf{y})\;\delta(\mathbf{y}-\mathbf{x})\Pi^\gamma_{\mathsf{GI}\,i}(\mathbf{x})\right\} = (-1)^n\frac{\partial}{\partial y_j}\int\frac{d\mathbf{z}(1)}{4\pi|\mathbf{y}-\mathbf{z}(1)|}A^{\delta_1}_{\mathsf{GI}\,l_1}(\mathbf{z}(1))\frac{\partial}{\partial z(1)_{l_1}}\times$$
$$\int\frac{d\mathbf{z}(2)}{4\pi|\mathbf{z}(1)-\mathbf{z}(2)|}A^{\delta_2}_{\mathsf{GI}\,l_2}(\mathbf{z}(2))\frac{\partial}{\partial z(2)_{l_2}}\cdots\int\frac{d\mathbf{z}(n)}{4\pi|\mathbf{z}(n-1))-\mathbf{z}(n)|}\times$$
$$A^{\delta_n}_{\mathsf{GI}\,l_n}(\mathbf{z}(n))\frac{\partial}{\partial z(n)_{l_n}}\frac{1}{4\pi|\mathbf{z}(n)-\mathbf{x}|}\Pi^\gamma_{\mathsf{GI}\,i}(\mathbf{x}) \qquad (69)$$

and the leading ($n=0$) term of the commutator given in Eq. (67) is

$$ig\epsilon^{\alpha\beta\gamma}\left(\frac{\partial}{\partial x_i}\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\Pi^\gamma_{\mathsf{GI}\,j}(\mathbf{y}) + \frac{\partial}{\partial y_j}\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\Pi^\gamma_{\mathsf{GI}\,i}(\mathbf{x})\right)\,.$$

The delta-functions that appear in Eq. (68) eliminate the integrations over the last of the inverse laplacian in the chain described in Eq. (56). When there is no delta-function in the expressions on which $\mathcal{T}^{\vec{\delta}}_{(n)}$ acts, the last inverse laplacian is also integrated over, as can be seen by comparing with Eqs. (55) and (57).

We can apply the same procedure used to obtain Eq. (68) to the commutator of $\Pi^{\alpha}_{\mathsf{GI}\,i}(\mathbf{x})$ and the hermitian adjoint of $\Pi^{\beta}_{\mathsf{GI}\,j}(\mathbf{y})$, in which case we get

$$\left[\Pi^{\alpha}_{\mathsf{GI}\,i}(\mathbf{x})\,,\Pi^{\beta\,\dagger}_{\mathsf{GI}\,j}(\mathbf{y})\right]=R_{\alpha a}(\mathbf{x})\Pi^{b}_{j}(\mathbf{y})\left[\Pi^{\alpha}_{i}(\mathbf{x})\,,R_{\beta b}(\mathbf{y})\right]-\left[\Pi^{\beta}_{\mathsf{GI}\,j}(\mathbf{y})\,,R_{\alpha a}(\mathbf{x})\right]\Pi^{a}_{i}(\mathbf{x})\,,\tag{70}$$

and therefore that

$$\begin{aligned}\left[\Pi^{\alpha}_{\mathsf{GI}\,i}(\mathbf{x})\,,\Pi^{\beta\,\dagger}_{\mathsf{GI}\,j}(\mathbf{y})\right]=i\sum_{n=0}g^{n+1}(-1)^{n+1}\,&\Big[\epsilon^{\vec{\delta}\alpha h}_{n}\epsilon^{h\gamma\beta}\Pi^{\gamma\,\dagger}_{\mathsf{GI}\,j}(\mathbf{y})\tfrac{\partial_i}{\partial^2}\left\{\mathcal{T}^{\vec{\delta}}_{(n)}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\right\}\\&-\;\epsilon^{\vec{\delta}\beta h}_{n}\epsilon^{h\gamma\alpha}\tfrac{\partial_j}{\partial^2}\left\{\mathcal{T}^{\vec{\delta}}_{(n)}(\mathbf{y})\delta(\mathbf{y}-\mathbf{x})\Pi^{\gamma}_{\mathsf{GI}\,i}(\mathbf{x})\right\}\Big]\\&-\left[R_{\beta b}(\mathbf{y})\,,\Pi^{a}_{i}(\mathbf{x})\right]\left[R_{\alpha a}(\mathbf{x})\,,\Pi^{b}_{j}(\mathbf{y})\right]\,.\end{aligned}\tag{71}$$

An alternate expression for Eq. (68) can be obtained by defining $\mathcal{D}(\mathbf{x},\mathbf{y})$ as an inverse of the Faddeev-Popov operator[7] $(D{\cdot}\partial)^{ab}=(\delta_{ab}+g\epsilon^{aub}A^{u}_{\mathsf{GI}\,n}\frac{\partial_n}{\partial^2})\partial^2$, given by

$$D{\cdot}\partial(\mathbf{x})\mathcal{D}(\mathbf{x},\mathbf{y})=\delta(\mathbf{x}-\mathbf{y}).\tag{72}$$

$\mathcal{D}(\mathbf{x},\mathbf{y})$ can be expanded as a series in the form

$$\mathcal{D}^{dh}(\mathbf{x},\mathbf{y})=-\frac{1}{\partial^2}\sum_{n=0}\epsilon^{\vec{\delta}dh}_{(n)}(-1)^{n}g^{n}\mathcal{T}^{\vec{\delta}}_{(n)}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\tag{73}$$

so that Eq. (68) can be expressed as

$$\left[\Pi^{\alpha}_{\mathsf{GI}\,i}(\mathbf{x})\,,\Pi^{\beta}_{\mathsf{GI}\,j}(\mathbf{y})\right]=ig\left\{\partial_i\mathcal{D}^{\alpha h}(\mathbf{x},\mathbf{y})\epsilon^{h\gamma\beta}\Pi^{\gamma}_{\mathsf{GI}\,j}(\mathbf{y})-\partial_j\mathcal{D}^{\beta h}(\mathbf{y},\mathbf{x})\epsilon^{h\gamma\alpha}\Pi^{\gamma}_{\mathsf{GI}\,i}(\mathbf{x})\right\}.\tag{74}$$

In the form given in Eq. (74), the equal-time commutation relation in Eq. (68) can be seen to be in agreement with the one given by Schwinger in Ref.[5] *modulo* the fact that Schwinger's Coulomb-gauge chromoelectric field operators are defined to be hermitian, whereas ours are not. The operator-ordering in the commutators in Ref. [5] that correspond to our Eq. (68) therefore differs from ours. We have defined the gauge-invariant chromoelectric fields so that the Hamiltonian in Eq. (51) has a simple and tractable form. Furthermore, with $D^{ba}_{i}=\delta_{ba}\partial_i+g\epsilon^{bua}A^{u}_{\mathsf{GI}\,i}$, Eq. (73), and the identity

$$\begin{aligned}&\left(\delta_{ij}\delta_{ba}+\frac{\partial_j}{\partial^2}\sum_{n=0}\epsilon^{\vec{\delta}bh}_{(n)}\,(-1)^{n}g^{n}\mathcal{T}^{\vec{\delta}}_{(n)}(\mathbf{x})D^{ha}_{i}(\mathbf{x})\right)\delta(\mathbf{x}-\mathbf{y})=\\&\qquad\sum_{n=0}\epsilon^{\vec{\delta}ba}_{(n)}(-1)^{n+1}g^{n}\frac{\partial_j}{\partial^2}\mathcal{T}^{\vec{\delta}}_{(n)\,k}(\mathbf{x})\left(\delta_{ik}-\frac{\partial_i\partial_k}{\partial^2}\right)\delta(\mathbf{x}-\mathbf{y})\end{aligned}\tag{75}$$

[7]In this, as well as in a number of similar cases, the gauge-invariant field $A^{c}_{\mathsf{GI}\,i}$ replaces the standard gauge-dependent temporal-gauge field A^{c}_{i} in the gauge-covariant derivative.

we can also confirm the agreement between Eq. (23) and the corresponding commutation rule given by Schwinger in Ref. [5], again *modulo* operator order in the corresponding expressions. Our commutators in Eqs. (23) and (68) differ much more substantially from those in Refs.[7, 8, 9] because, in these references, the Coulomb-gauge chromoelectric field is defined to be transverse, in contrast to the practice followed in Schwinger's treatment of the Coulomb gauge as well as in our work.

We can use Eq. (65) to substitute explicit expressions for the commutators in the last line of Eq. (71); but comparison of Eqs. (68) and (71) suffices to confirm that $\Pi^{\beta\,\dagger}_{\mathsf{GI}\,j}(\mathbf{y})$ differs from $\Pi^{\beta}_{\mathsf{GI}\,j}(\mathbf{y})$. It is also possible to evaluate $\Pi^{\beta\,\dagger}_{\mathsf{GI}\,j}(\mathbf{y}) - \Pi^{\beta}_{\mathsf{GI}\,j}(\mathbf{y})$ directly. The resulting expression is clearly nonvanishing, but the expression is lengthy as well as singular and it is not necessary to quote it in detail in order to make the point that the gauge-invariant momentum (and negative chromoelectric field) is not hermitian. We observe that $\partial_i\Pi^a_{\mathsf{GI}\,i}$ and $g\epsilon^{abc}A^b_{\mathsf{GI}\,i}\Pi^b_{\mathsf{GI}\,i}+j^a_{0\,\mathsf{GI}}$, the two component parts of $\hat{\mathcal{G}}^a_{\mathsf{GI}}$, are separately gauge-invariant, but are not separately hermitian.

It is also of interest to examine whether the commutator algebra of the gauge-invariant Gauss's law operators closes, by examining the commutator $[\hat{\mathcal{G}}^a_{\mathsf{GI}}(\mathbf{x}),\, \hat{\mathcal{G}}^b_{\mathsf{GI}}(\mathbf{y})]$. From Eq. (44), we observe that

$$\left[\mathcal{G}^a_{\mathsf{GI}}(\mathbf{x}),\, \mathcal{G}^b_{\mathsf{GI}}(\mathbf{y})\right] = \mathcal{R}_{ac}(\mathbf{x})\left[\mathcal{G}^c(\mathbf{x}),\, \mathcal{R}_{bd}(\mathbf{y})\right] - \mathcal{R}_{bd}(\mathbf{y})\left[\mathcal{G}^d(\mathbf{y}),\, \mathcal{R}_{ac}(\mathbf{x})\right] + \mathcal{R}_{ac}(\mathbf{x})\mathcal{R}_{bd}(\mathbf{y})\left[\mathcal{G}^c(\mathbf{x}),\, \mathcal{G}^d(\mathbf{y})\right] \tag{76}$$

and, using

$$[\mathcal{G}^c(\mathbf{x}),\, \mathcal{R}_{bd}(\mathbf{y})] = ig\epsilon^{cdq}\mathcal{R}_{bq}(\mathbf{y})\delta(\mathbf{x}-\mathbf{y}), \tag{77}$$

$$\text{that} \quad \left[\mathcal{G}^a_{\mathsf{GI}}(\mathbf{x}),\, \mathcal{G}^b_{\mathsf{GI}}(\mathbf{y})\right] = -ig\epsilon^{abc}\mathcal{G}^c_{\mathsf{GI}}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}). \tag{78}$$

This shows that the commutator algebra of the gauge-invariant Gauss's law operators closes and, in fact, is almost identical to the algebra of the gauge dependent Gauss's law operators except for a relative sign change between the two cases.

One consequence of the time-independence of the non-Abelian Gauss's law operator given in Eq. (1) is the fact that $[H, \mathcal{G}^a(\mathbf{x})] = 0$, so that when a state implements Gauss's law (*i.e.* when $\mathcal{G}^a(\mathbf{x})\,|\Psi\rangle = 0$), the time-evolved state $\mathcal{G}^a(\mathbf{x})\exp(-iHt)|\Psi\rangle$ also vanishes. It is therefore important to inquire whether a similar property can be ascribed to the gauge-invariant non-Abelian Gauss's law operator $\mathcal{G}^a_{\mathsf{GI}}$, so that the set of states that implement $\mathcal{G}^a_{\mathsf{GI}}(\mathbf{x})\,|\Psi\rangle = 0$ also implement that same constraint at all later times, when $\exp(-i\hat{H}_{\mathsf{GI}}t)$ is the time evolution operator. We address this question by observing that

$$\left[\mathcal{G}^a_{\mathsf{GI}}(\mathbf{x}),\, \Pi^b_{\mathsf{GI}}(\mathbf{y})\right] = \sum_{r=0}(-1)^{r+1}g^{r+1}\epsilon^{\vec{v}bh}_{r}\epsilon^{hqa}\tfrac{\partial_j}{\partial^2}\left(\mathcal{T}^{\vec{v}}_{(r)}(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})\right)\mathcal{G}^q_{\mathsf{GI}}(\mathbf{x}) \tag{79}$$

follows from Eqs. (66) and (44), and that $\mathcal{G}^a_{\mathsf{GI}}(\mathbf{x})$ commutes with $\mathcal{K}^b_0(\mathbf{r})$ and with $j^b_0(\mathbf{r})$. The commutator of $\hat{H}_{\mathsf{GI}}$ and $\mathcal{G}^a_{\mathsf{GI}}(\mathbf{x})$ therefore receives contributions only from commutators of

$\Pi^q_{\mathsf{GI}}(\mathbf{y})$ and $\mathcal{G}^a_{\mathsf{GI}}(\mathbf{x})$; and, when $[\Pi^c_{\mathsf{GI}}(\mathbf{y}),\,\mathcal{G}^a_{\mathsf{GI}}(\mathbf{x})]$ is to the left of another operator, as in $\int d\mathbf{y}[\Pi^c_{\mathsf{GI}}(\mathbf{y}),\,\mathcal{G}^a_{\mathsf{GI}}(\mathbf{x})]\Pi^c_{\mathsf{GI}}(\mathbf{y})$ or in $\int d\mathbf{y} g\epsilon^{bpq}A^p_{\mathsf{GI}}(\mathbf{y})[\Pi^q_{\mathsf{GI}}(\mathbf{y}),\,\mathcal{G}^a_{\mathsf{GI}}(\mathbf{x})](\partial^{-2})\mathcal{K}^b_0(\mathbf{y})$, the resulting $\mathcal{G}^a_{\mathsf{GI}}(\mathbf{x})$ either commutes with the operators to its right, or, in moving to the right of those that do not commute with it, produces still other terms that have a $\mathcal{G}^d_{\mathsf{GI}}(\mathbf{r})$ on the extreme right-hand side. We therefore observe that the most general expression for the commutator of $\hat{H}_{\mathsf{GI}}$ and $\mathcal{G}^a_{\mathsf{GI}}(\mathbf{x})$ is

$$\left[\hat{H}_{\mathsf{GI}},\,\mathcal{G}^a_{\mathsf{GI}}(\mathbf{x})\right] = \int d\mathbf{r}\chi^{ac}(\mathbf{r},\mathbf{x})\mathcal{G}^c_{\mathsf{GI}}(\mathbf{r}) \tag{80}$$

where $\chi^{ac}(\mathbf{r},\mathbf{x})$ is a nonlocal functional of $\mathbf{x}$ and $\mathbf{r}$. Although $\mathcal{G}^a_{\mathsf{GI}}(\mathbf{x})$ is not time-independent, it is nevertheless true that a state $|\Psi\rangle$ for which $\mathcal{G}^a_{\mathsf{GI}}(\mathbf{x})\,|\Psi\rangle = 0$ for all values of $\mathbf{x}$ and a will time-evolve so that $\mathcal{G}^a_{\mathsf{GI}}(\mathbf{x})\,\exp(-i\hat{H}_{\mathsf{GI}}\,t)|\Psi\rangle = 0$ as well.

The Hamiltonian given in Eqs. (50)-(59) relates QCD in the temporal-gauge to its Coulomb-gauge formulation. $\hat{H}_{\mathsf{GI}}$ is the Hamiltonian of QCD in the temporal gauge. It is expressed in terms of gauge-invariant fields, and is displayed in these equations in a representation obtained in Ref.[2] by unitarily transforming the standard form of that Hamiltonian. But that does not change the fact that $\hat{H}_{\mathsf{GI}}$ is still the temporal-gauge Hamiltonian. In principle, $\hat{H}_{\mathsf{GI}}$ could be used to derive the equations of motion of both, the gauge-invariant and the standard gauge-dependent temporal gauge fields, although the calculation leading to the latter would be clumsy and exceedingly tedious in this representation. In such calculations, we could not neglect the contributions made by commutators of operator-valued fields with $H_{\mathcal{G}}$, since the latter is an indispensable part of the temporal-gauge Hamiltonian in this particular representation. In reference to Eqs. (50)-(59), H_{GI} and $H_{\mathcal{G}}$ — the two constituent parts of $\hat{H}_{\mathsf{GI}}$ — can be given the following interpretation: H_{GI} can be recognized as a representation of the QCD Hamiltonian in the Coulomb gauge, similar to the Coulomb-gauge Hamiltonians obtained by methods different from ours. [5, 6, 7, 8, 9] These works differ among themselves and from ours in various ways — in operator order, in the inclusion or omission of the longitudinal component of the chromoelectric field, in whether the chromoelectric field and the glue color-charge density are hermitian — but the Hamiltonians given by these authors are markedly similar to our H_{GI}. Furthermore, $A^a_{\mathsf{GI}\,i}$ and $\Pi^d_{\mathsf{GI}\,j}$ respectively have the same commutation rules with each other, and the components of $\Pi^d_{\mathsf{GI}\,j}$ among themselves, as do the corresponding Coulomb-gauge fields in Ref. [5], *modulo* operator ordering.

$H_{\mathcal{G}}$ has no role in the time evolution of physical systems, since it annihilates all states that implement Gauss's law (defined by $\mathcal{G}^a_{\mathsf{GI}}(\mathbf{r})\,|\Psi\rangle = 0$). When $\mathcal{G}^a_{\mathsf{GI}}$ annihilates a state $|\Psi\rangle$, and thus identifies it as implementing Gauss's law, it also annihilates $\exp(-i\hat{H}_{\mathsf{GI}}t)|\Psi\rangle$. As we have shown, if $H_{\mathcal{G}}$ appears within a sequence of operators constructed from component parts of $\hat{H}_{\mathsf{GI}}$, such as might be found in an expansion of $\exp(-i\hat{H}_{\mathsf{GI}}t)$ or in a transition amplitude, and if this operator sequence acts on a state $|\Psi\rangle$ that satisfies $\mathcal{G}^a_{\mathsf{GI}}(\mathbf{x})|\Psi\rangle = 0$, then the resulting expression must vanish. $\mathcal{G}^a_{\mathsf{GI}}$ can be commuted through the other operator to its right, producing further operators, each of which has a $\mathcal{G}^a_{\mathsf{GI}}$ on its right-hand

side, until, finally all resulting terms have a $\mathcal{G}^a_{\mathsf{GI}}$ operator for some value of a and some spatial argument, acting on $|\Psi\rangle$, with the result that the original expression (*i.e.* the sequence of operators acting on $|\Psi\rangle$) vanishes. Finally, we note that $\hat{H}_{\mathsf{GI}}$ is not affected by the operator-ordering problem associated with the direct quantization of QCD in the Coulomb gauge referred to in Ref. [7]. The gauge-invariant field and negative chromoelectric field, $A^a_{\mathsf{GI}\,i}$ and $\Pi^b_{\mathsf{GI}\,j}$ respectively, obey commutation rules derived from their structure in terms of standard gauge-dependent temporal-gauge fields, whose commutation rules are simple enough to make the commutation relations of $A^a_{\mathsf{GI}\,i}$ and $\Pi^b_{\mathsf{GI}\,j}$, as well as their positions in the QCD Hamiltonian, well-defined;

A remarkably similar state of affairs obtains in QED. When QED is formulated in the temporal gauge, and a unitary transformation is carried out which is the Abelian analog to the one that takes QCD in the temporal gauge from the $\mathcal{C}$ to the $\mathcal{N}$ transformation as discussed in Ref. [1], the following result is obtained:[26, 27] The QED Hamiltonian in the temporal gauge, unitarily transformed by the Dirac transformation,[20] can be described as

$$\hat{H}_{QED} = \int d\mathbf{r}\,\left[\tfrac{1}{2}\Pi_i(\mathbf{r})\Pi_i(\mathbf{r}) + \tfrac{1}{4}F_{ij}(\mathbf{r})F_{ij}(\mathbf{r}) + \psi^\dagger(\mathbf{r})\left(\beta m - i\alpha_i\partial_i\right)\psi(\mathbf{r})\right] \\ - \int d\mathbf{r} A_i^{(T)}(\mathbf{r})j_i(\mathbf{r}) + \int d\mathbf{r}d\mathbf{r}'\frac{j_0(\mathbf{r})j_0(\mathbf{r}')}{8\pi|\mathbf{r}-\mathbf{r}'|} + H_g \tag{81}$$

where $A_i^{(T)}$ designates the transverse Abelian gauge field, and H_g can be expressed as

$$H_g = -\tfrac{1}{2}\int d\mathbf{r}\left(\partial_i\Pi_i(\mathbf{r})\frac{1}{\partial^2}j_0(\mathbf{r}) + j_0(\mathbf{r})\frac{1}{\partial^2}\partial_i\Pi_i(\mathbf{r})\right)\,, \tag{82}$$

so that all the operator-valued fields that appear in $\hat{H}_{QED}$ are gauge invariant, and so that $\hat{H}_{QED}$ also consists of two parts: the Hamiltonian for QED in the Coulomb gauge and H_g, which has no affect on the time evolution of states that implement Gauss's law (which, in analogy to the non-Abelian $\mathcal{G}^a_{\mathsf{GI}}(\mathbf{r})\,|\Psi\rangle = 0$, takes the form $\partial_i\Pi_i(\mathbf{r})|\Phi\rangle = 0$ after the Dirac transformation has been carried out).[8] When the time derivative is defined as $i[\hat{H}_{QED}\,,]$, the equations of motion of temporal-gauge QED result. But since H_g can have no effect on the time evolution of state vectors that implement Gauss's law, the only part of $\hat{H}_{QED}$ that time-evolves such state vectors is the Coulomb-gauge Hamiltonian. This remarkable parallelism between QCD and QED prevails because $V_{\mathcal{C}}$, which we constructed in Ref.[1], is the non-Abelian generalization of the transformation which Dirac constructed for QED.

[8] In Refs. [26, 27], $\partial_i\Pi_i$ is separated into positive and negative frequency parts, so that the gauge-invariant charged states is manifestly normalizable. The unitary transformation in these references therefore differs somewhat from the one introduced by Dirac, but has essentially the same effect.

3 Multiple solutions of the resolvent field equations and Gribov copies

The fact that we can identify the Coulomb-gauge field with the gauge-invariant fields we have constructed,[1] enables us to use previously reported results about the topology of the gauge-invariant gauge field [4] to observe how the Gribov ambiguity manifests itself in the temporal-gauge formulation we have been discussing.

The Gribov ambiguity — the existence of multiple copies of fields that obey the Coulomb gauge condition — complicates the quantization of QCD in the Coulomb gauge. This complication manifests itself as an inability to uniquely invert the Faddeev-Popov operator $D_i\partial_i$. Gauss's law in the Coulomb gauge can be written in the form $D_i\partial_i A_0^a = -j_0^a$, in which the Faddeev-Popov operator takes account of the fact that the "total" color charge-density (including quark and glue color contributions) is $J_0^a = g\epsilon^{abc}A_i^b\Pi_i^c + j_0^a$. It is necessary to invert $D_i\partial_i A_0^a = -j_0^a$ in order to replace A_0^a in the Coulomb-gauge Hamiltonian with its equivalent in terms of unconstrained field variables. This inversion produces not only operator-ordering problems; it also leads to the Gribov problem. Inverting $D_i\partial_i A_0^a = -j_0^a$ formally produces the equation $A_0^a = -(D_i\partial_i)^{-1}j_0^a$, the non-Abelian analog of $A_0 = -\partial^{-2}j_0$ in QED. However, whereas ∂^2 is trivially invertible for suitably chosen boundary conditions, and manifestly commutes with the Abelian gauge field as well as with its conjugate momentum, the differential operator $D_i\partial_i$ is not similarly invertible and does not commute with the non-Abelian gauge field or with the chromoelectric field. A_0^a therefore cannot be uniquely replaced by an expression that depends on unconstrained field variables only.

Various authors have responded to this complication in different ways. Some have concluded that the difficulties associated with the Coulomb gauge make it preferable to use an axial gauge, such as the temporal gauge, in which, they state, the Gribov problem does not exist.[28] Other authors have taken the point of view that, since it is likely that Gribov copies of the original gauge field belong to different topological sectors, one can ignore the Gribov copies by limiting oneself to the perturbative regime. [29] It has also been questioned whether the Gribov ambiguity is truly absent when axial gauges are imposed, or whether it always affects non-Abelian gauge theories, but is more obscure and harder to detect in the case of axial gauges. Singer has proven that when the path integral of a non-Abelian gauge theory is defined over a Euclidean 4-dimensional sphere, ambiguities appear regardless of the gauge condition; but, as Singer has noted, in the absence of such boundary conditions, there are gauges — in particular axial gauges — to which his proof does not apply.[30] Others have argued that given these circumstances, it is still uncertain whether Gribov ambiguities are endemic to non-Abelian gauge theories or whether they only mark the Coulomb gauge as an unfortunate choice of gauge.[29]

In previous work[1, 2, 3, 4] and in previous sections of this paper, we have given series representations of the resolvent field as well as of the inverse of the Faddeev-Popov

operator and of a number of Dirac commutators. We have also shown that even when functionals of gauge fields are given a series representation, that does not necessarily eliminate the possibility of representing them nonperturbatively. These functionals can be related by integral equations, and nonperturbative solutions of these equations can be obtained. In this way, the origin of multiple solutions of gauge fields can be better understood

In Ref.[4], we made an *ansatz*, representing the gauge field in the temporal gauge and the resolvent field as functions of spatial variables that are second-rank tensors in the combined spatial and SU(2) indices; except in so far as the forms of $\overline{\mathcal{A}_i^\gamma}(\mathbf{r})$ and $A_i^\gamma(\mathbf{r})$ reflect this second-rank tensor structure, they are represented as isotropic functions of position. In this way, we can represent the longitudinal part of the gauge field in the temporal gauge as

$$A_i^{\gamma\,L}(\mathbf{r}) = \frac{1}{g}\left[\delta_{i\gamma}\,\frac{\mathcal{N}(r)}{r} + \frac{r_i\,r_\gamma}{r}\,\left(\frac{\mathcal{N}(r)}{r}\right)'\,\right] \tag{83}$$

and the transverse part as

$$A_i^{\gamma\,T}(\mathbf{r}) = \delta_{i\gamma}\,\mathcal{T}_A(r) + \frac{r_i\,r_\gamma}{r^2}\,\mathcal{T}_B(r) + \epsilon_{i\gamma n}\frac{r_n}{r}\,\mathcal{T}_C(r) \tag{84}$$

where $\mathcal{N}(r)$, $\mathcal{T}_A(r)$, $\mathcal{T}_B(r)$ and $\mathcal{T}_C(r)$ are isotropic functions of r, the prime denotes differentiation with respect to r, and the transversality of $A_i^{\gamma\,T}(\mathbf{r})$ requires that

$$\frac{d(r^2\mathcal{T}_B)}{dr} + r^2\frac{d\,\mathcal{T}_A}{dr} = 0\,. \tag{85}$$

An entirely analogous representation of the resolvent field enables us to relate it to the gauge field through the nonlinear integral equation described in Section 1. As a result of this analysis, we have been able to show that it is possible to represent the resolvent field as [4]

$$\overline{\mathcal{A}_i^\gamma}(\mathbf{r}) = \left(\delta_{i\gamma} - \frac{r_i\,r_\gamma}{r^2}\right)\left(\frac{\overline{\mathcal{N}}}{gr} + \varphi_A\right) + \epsilon_{i\gamma n}\frac{r_n}{r}\,\varphi_C \tag{86}$$

where

$$\varphi_A = \frac{1}{gr}\left[\mathcal{N}\cos(\overline{\mathcal{N}}+\mathcal{N}) - \sin(\overline{\mathcal{N}}+\mathcal{N})\right] + \mathcal{T}_A\left[\cos(\overline{\mathcal{N}}+\mathcal{N}) - 1\right] - \mathcal{T}_C\sin(\overline{\mathcal{N}}+\mathcal{N}) \tag{87}$$

and

$$\varphi_C = \frac{1}{gr}\left[\mathcal{N}\sin(\overline{\mathcal{N}}+\mathcal{N}) + \cos(\overline{\mathcal{N}}+\mathcal{N}) - 1\right] + \mathcal{T}_C\left[\cos(\overline{\mathcal{N}}+\mathcal{N}) - 1\right] + \mathcal{T}_A\sin(\overline{\mathcal{N}}+\mathcal{N}). \tag{88}$$

Similarly, the gauge-invariant gauge field can be expressed as a functional of $\overline{\mathcal{N}}$ and of $\mathcal{N}$, $\mathcal{T}_A$, $\mathcal{T}_B$ and $\mathcal{T}_C$ as shown by

$$
\begin{aligned}
&A^{\gamma}_{\mathsf{GI}\,i}(\mathbf{r}) = \frac{1}{gr}\left\{\epsilon_{i\,\gamma\,n}\frac{r_n}{r}\left[\cos(\overline{\mathcal{N}}+\mathcal{N}) - 1 + \mathcal{N}\sin(\overline{\mathcal{N}}+\mathcal{N})\right] + \left(\delta_{i\,\gamma} - \frac{r_i r_\gamma}{r^2}\right)\times\right.\\
&\left.\times\left[\mathcal{N}\cos(\overline{\mathcal{N}}+\mathcal{N}) - \sin(\overline{\mathcal{N}}+\mathcal{N})\right] - \frac{r_i r_\gamma}{r}\frac{d\overline{\mathcal{N}}}{dr}\right\}\\
&+\mathcal{T}_A\left\{\left(\delta_{i\,\gamma} - \frac{r_i r_\gamma}{r^2}\right)\cos(\overline{\mathcal{N}}+\mathcal{N}) + \epsilon_{i\,\gamma\,n}\frac{r_n}{r}\sin(\overline{\mathcal{N}}+\mathcal{N})\right\} + \frac{r_i r_\gamma}{r^2}\left(\mathcal{T}_A + \mathcal{T}_B\right)\\
&+\mathcal{T}_C\left\{\epsilon_{i\,\gamma\,n}\frac{r_n}{r}\cos(\overline{\mathcal{N}}+\mathcal{N}) - \left(\delta_{i\,\gamma} - \frac{r_i r_\gamma}{r^2}\right)\sin(\overline{\mathcal{N}}+\mathcal{N})\right\}.
\end{aligned}
\tag{89}
$$

With these representations, the nonlinear integral equation that relates the resolvent field to the gauge field (in the temporal-gauge) is transformed to the nonlinear differential equation

$$
\begin{aligned}
\frac{d^2\overline{\mathcal{N}}}{du^2} \;&+\; \frac{d\overline{\mathcal{N}}}{du} + 2\left[\mathcal{N}\cos(\overline{\mathcal{N}}+\mathcal{N}) - \sin(\overline{\mathcal{N}}+\mathcal{N})\right]\\
&+\; 2gr_0\exp(u)\left\{\mathcal{T}_A\left[\cos(\overline{\mathcal{N}}+\mathcal{N}) - 1\right] - \mathcal{T}_C\sin(\overline{\mathcal{N}}+\mathcal{N})\right\} = 0
\end{aligned}
\tag{90}
$$

where $u = \ln(r/r_0)$ and r_0 is an arbitrary constant. $\overline{\mathcal{N}}$ — the dependent variable in Eq. (90) — can be directly related to the resolvent field $\overline{\mathcal{A}^{\gamma}_i}$ by

$$
\overline{\mathcal{N}} = g\frac{r_\alpha}{r}\frac{\partial_j}{\partial^2}\overline{\mathcal{A}^{\alpha}_j}(\mathbf{r}). \tag{91}
$$

Similarly, $\mathcal{N}$ can be related to the gauge field in the temporal (Weyl) gauge as shown by

$$
\mathcal{N} = g\frac{r_\alpha}{r}\frac{\partial_j}{\partial^2}A^{\alpha}_j(\mathbf{r}). \tag{92}
$$

$\mathcal{T}_A$, $\mathcal{T}_B$ and $\mathcal{T}_C$ are determined by Eq. (85) and by the transverse part of the gauge field as shown by

$$
\frac{r_i r_\gamma}{r^2}A^{\gamma\,T}_i = \mathcal{T}_A + \mathcal{T}_B \quad\text{and by}\quad \frac{1}{2}\epsilon^{i\gamma j}\frac{r_j}{r}A^{\gamma\,T}_i = \mathcal{T}_C\,. \tag{93}
$$

In solving Eq. (90), boundary conditions are imposed on $\overline{\mathcal{N}}$ and on the source terms $\mathcal{N}$, $\mathcal{T}_A$, $\mathcal{T}_B$ and $\mathcal{T}_C$. $\mathcal{T}_A$, $\mathcal{T}_B$ and $\mathcal{T}_C$ are required to vanish as $u \to \pm\infty$; $\mathcal{N}$ is required to vanish as $u\to-\infty$, and to either vanish or approach specified limits as $u\to\infty$. $\overline{\mathcal{N}}$ is required to be bounded in the entire interval $-\infty\leq u\leq\infty$. For convenience, we normalize $\overline{\mathcal{N}}$ so that $\overline{\mathcal{N}}\to 0$ when $u\to-\infty$.

One of the questions addressed in Ref.[4] is the following: given a specified set of values $\mathcal{N}(r)$, $\mathcal{T}_A(r)$, $\mathcal{T}_B(r)$ and $\mathcal{T}_C(r)$, what behavior is possible for $\overline{\mathcal{N}}$, if $\overline{\mathcal{N}}$ is required to be bounded in the entire interval $0\leq r<\infty$ and to obey Eq. (90)? In posing this question,

$\overline{\mathcal{N}}$ is seen to be the unknown function, and $\mathcal{N}(r)$, $\mathcal{T}_A(r)$, $\mathcal{T}_B(r)$ and $\mathcal{T}_C(r)$ are source terms that drive the solution, subject to the required boundary conditions. In Ref. [4], we presented numerical integrations of Eq. (90) which demonstrated that a number of different solutions of this equation — and, hence, a number of different $A_{\mathsf{GI}\,i}^{\gamma}$ — can be based on the same gauge-dependent gauge field characterized by a single set of values of $\mathcal{N}$, $\mathcal{T}_A$, $\mathcal{T}_B$ and $\mathcal{T}_C$. A related question — what form for $A_{\mathsf{GI}\,i}^{\gamma}(\mathbf{r})$ corresponds to the different possible functional forms of $\overline{\mathcal{N}}$ — can also be posed. When the gauge field $A_i^{\gamma}=0$, the resolvent field $\bar{\mathcal{A}}_i^{\gamma}$ and the gauge-invariant field $A_{\mathsf{GI}\,i}^{\gamma}$ need not vanish. In that case, Eq. (90) reduces to the autonomous[9] "Gribov equation" [11, 12]

$$\frac{d^2\overline{\mathcal{N}}}{du^2}+\frac{d\overline{\mathcal{N}}}{du}-2\sin(\overline{\mathcal{N}})=0 \tag{94}$$

which is also the equation for a damped pendulum with $\overline{\mathcal{N}}$ representing the angle with respect to the pendulum's position of unstable equilibrium, and u representing the time.[10] In the application of this equation to the Gribov problem, $\overline{\mathcal{N}}$ must remain bounded not only in the interval $0\leq u<\infty$, but also in the larger interval $-\infty<u<\infty$ to include the entire configuration space $0\leq r<\infty$.

The restriction that $\overline{\mathcal{N}}$ must be bounded in the entire interval $(-\infty\leq u\leq\infty)$ severely limits the allowed solutions of Eq. (94). For solutions of Eq. (94) that are bounded in the entire interval $(-\infty\leq u\leq\infty)$, there is a single, unique, phase plot. One branch extends from the unstable saddle point (corresponding to $u\rightarrow-\infty$), at which $\overline{\mathcal{N}}=0$, to a stable point (corresponding to $u\rightarrow\infty$), at which $\overline{\mathcal{N}}=\pi$. The other branch extends from the same saddle point (at which u corresponds to $-\infty$) to a stable point at $\overline{\mathcal{N}}=-\pi$, also corresponding to $u\rightarrow\infty$. The two branches are identical, except that for each point on one branch the values of $\overline{\mathcal{N}}$ and $d\overline{\mathcal{N}}/du$ correspond to values $-\overline{\mathcal{N}}$ and $-d\overline{\mathcal{N}}/du$ on the other. Numerical solutions are obtained by setting $\overline{\mathcal{N}}$ and $d\overline{\mathcal{N}}/du$ equal to the same small value at some large negative value of u, in order to discriminate against solutions that become unbounded as u approaches the saddle point as $u\rightarrow-\infty$, as discussed in Ref.[4]. We can choose different large negative values of u at which to set $\overline{\mathcal{N}}=d\overline{\mathcal{N}}/du=0$; if, in one case, we choose u_a and in another u_b, we obtain solutions $\overline{\mathcal{N}}_a(u)$ and $\overline{\mathcal{N}}_b(u)$ respectively, where $\overline{\mathcal{N}}_b(u)=\overline{\mathcal{N}}_a(u+\mathsf{U})$ and $\mathsf{U}=u_b-u_a$. Similarly, changing the magnitude of the small value of $\overline{\mathcal{N}}(u_a)=d\overline{\mathcal{N}}(u_a)/du=\epsilon$ to $\overline{\mathcal{N}}(u_a)=d\overline{\mathcal{N}}(u_a)/du=\epsilon'$ also has the effect of shifting the functional form of $\overline{\mathcal{N}}(u)$ from u to $u+u_0$ for a particular value of u_0. Replacing the initial condition $\overline{\mathcal{N}}(u_a)=d\overline{\mathcal{N}}(u_a)/du=\epsilon$, for a small ϵ, with $\overline{\mathcal{N}}(u_a)=d\overline{\mathcal{N}}(u_a)/du=-\epsilon$, has no other effect than changing the signs of $\overline{\mathcal{N}}(u)$ and $d\overline{\mathcal{N}}/du$. Figure 1 illustrates some of these relationships.

To study the variation in the form of $\overline{\mathcal{N}}$ that is allowed when $\overline{\mathcal{N}}$ is represented in configuration space in the form $\overline{\mathcal{N}}(u(r))=\overline{\mathcal{N}}(\ln(r/r_0))$, we observe the following: For the

[9] Eq. (94) is autonomous because the variable u does not appear explicitly in this equation; it appears implicitly only as the argument of $\overline{\mathcal{N}}$ and of its derivatives.

[10] the function α in Ref.[12] is related to $\overline{\mathcal{N}}$ by $\overline{\mathcal{N}}=2\alpha$.

shift $\overline{\mathcal{N}}_b(u) = \overline{\mathcal{N}}_a(u + \mathsf{U})$ with U represented as $\mathsf{U} = \ln(r_0/R_0)$ for an appropriate R_0,

$$\overline{\mathcal{N}}_b\left(\ln(r/r_0)\right) = \overline{\mathcal{N}}_a\left(\ln(r/r_0) + \ln(r_0/R_0)\right) = \overline{\mathcal{N}}_a\left(\ln(r/R_0)\right). \tag{95}$$

It is clear that when $\overline{\mathcal{N}}$ is required to be bounded in the entire interval $(0 \leq r \leq \infty)$, a change in the constant r_0 and an overall change of sign are the only changes allowed in $\overline{\mathcal{N}}\left(\ln(r/r_0)\right)$. In particular, in *every* such case, $\overline{\mathcal{N}}(r = 0) = 0$ and $\overline{\mathcal{N}}(r \to \infty) = \pm\pi$. When the restrictions on $\mathcal{N}$, $\mathcal{T}_A$, $\mathcal{T}_B$ and $\mathcal{T}_C$ that led to Eq. (94) are applied to Eq. (89), we find that

$$[A^{\gamma}_{\mathsf{GI}\,i}(\mathbf{r})]_{(0)} = \frac{-2}{gr}\epsilon_{i\,\gamma\,n}\frac{r_n}{r}, \tag{96}$$

where the subscript (0) designates the solution that corresponds to the "pure gauge" case in which the gauge-dependent gauge field A^{γ}_i vanishes. $[A^{\gamma}_{\mathsf{GI}\,i}(\mathbf{r})]_{(0)}$ can easily be recognized as a "hedgehog" solution.

Some authors have suggested that Eq. (94) — the Gribov equation — has a variety of solutions for which $\overline{\mathcal{N}}(r \to \infty)$ can be any integer multiple of π, [32] and that different integer multiples correspond to different topological sectors connected by large gauge transformations. This suggestion, which is motivated by the model of a damped pendulum, neglects the fact that in the Gribov equation u must be bounded in the entire interval $(-\infty \leq u \leq \infty)$, and that the only solutions of the damped pendulum problem that can remain bounded in this entire interval as the time u is extrapolated backwards to $-\infty$, are those for which the pendulum initially is at rest in its unstable equilibrium position. And, with this initial position, the damped pendulum is unable to execute multiple turns before coming to rest at equilibrium in a stable configuration.

Gribov explicitly noted that the necessity of requiring the solutions to Eq. (94) to be bounded in the entire interval $-\infty < u < \infty$ limits the asymptotic values of $\overline{\mathcal{N}}(u \to \infty)$ to $\pm\pi$.[12] He therefore also considered the transverse gauge field that results when an arbitrarily chosen transverse gauge field is gauge-transformed, so that the new field is not pure gauge, but is given by

$$\mathsf{A}'_i = U\mathsf{A}_i U^{-1} + iU\partial_i U^{-1} \tag{97}$$

$$\text{where } \; U = \exp\left(-i\phi(r)\frac{\vec{r}\cdot\vec{\tau}}{2r}\right) \;\text{ and }\; \mathsf{A}_i = \mathsf{A}^c_i\frac{\tau^c}{2} \;\text{ with }\; \mathsf{A}^c_i = \epsilon^{ijc}\frac{r_j}{r^2}f(r).$$

A'_i and A_i both are transverse and therefore belong to the Coulomb gauge. The transversality of A'_i leads to the equation

$$\frac{d^2\,\phi}{du^2} + \frac{d\,\phi}{du} - 2\sin(\phi)\left(1 - f(u)\right) = 0 \tag{98}$$

which is not autonomous, and does have the multiple solutions that correspond to Gribov copies belonging to different topological sectors.

In our work, Eq. (90) — of which Eq. (94) is a special case — also has inhomogeneous source terms that prevent it from being autonomous. But, unlike Eq. (98), Eq. (90) is not obtained by gauge-transforming from one Coulomb-gauge field to another. It is a consequence of the transformation from "standard" gauge-dependent fields in the temporal gauge to the corresponding gauge-invariant fields. Our Eq. (94) describes this transformation for the case that the initial gauge-dependent temporal-gauge field is set $= 0$, which clearly corresponds to the "pure gauge" case — Gribov's Eq. (98) with $f(u) = 0$. But, more generally, the relation between our Eqs. (90) and (94) is different from the relation of Gribov's Eq. (98) to that same equation with $f(u) = 0$. Our Eq. (90) would not reduce to an equation of the form of Eq. (98) even if the initial gauge-dependent field were chosen to be purely transverse. Eq. (90) shows that, in that case, there would be the additional "source"-term $2gr_0 \exp(u)\mathcal{T}_A(\cos\overline{\mathcal{N}} - 1)$ (where $\overline{\mathcal{N}}$ corresponds to ϕ in Eq. (98)). Nevertheless, in spite of these differences, the fact that we are able to identify the gauge-invariant gauge fields we constructed with the Coulomb-gauge fields enables us to interpret the multiple solutions of Eq. (90), which we demonstrated in Ref. [4], as Gribov copies. These multiple solutions represent Gribov copies of the gauge-invariant gauge fields in the temporal gauge, which manifest themselves even when there are no such copies, and no ambiguity, in the case of the "standard" gauge-dependent temporal-gauge fields.

4 Discussion

In this work, we have established a relationship between the gauge-invariant temporal-gauge and the Coulomb-gauge formulations of two-color QCD: The temporal-gauge QCD Hamiltonian, when represented entirely in terms of gauge-invariant operator-valued fields, is not identical to the Coulomb-gauge Hamiltonian represented by H_{GI} given in Eq. (51). But the two are physically equivalent — *i. e.* they lead to identical values of observable quantities within the space of states in which Gauss's law has been implemented. Moreover, we have shown that the gauge-invariant temporal-gauge fields obey commutation rules that are the same as those for Coulomb-gauge fields, *modulo* operator-ordering ambiguities, when the longitudinal components of the Coulomb-gauge chromoelectric field are retained, as they are in our work and in Schwinger's treatment of the Coulomb gauge.[5]

We have also shown, through reference to specific numerical calculations,[4] that there are Gribov copies of *gauge-invariant* fields in the temporal-gauge formulation of QCD, even though there are no Gribov copies of the *gauge-dependent* temporal-gauge fields. We have demonstrated that the gauge-invariant fields in the temporal gauge obey a nonlinear integral equation, which — subject to an *ansatz* — can be transformed to a nonlinear differential equation that has multiple solutions corresponding to a single gauge-dependent field. These solutions must be bounded in the entire configuration space. Thus, the nonlinear differential equation that embodies the imposition of Gauss's law and the implemen-

tation of gauge invariance in the temporal gauge, and the requirement of boundedness, lead to the multiple solutions that can be identified as Gribov copies. This is consistent with our demonstration of a close resemblance between the gauge-invariant formulation of the temporal gauge and the Coulomb-gauge formulation of QCD. It is also consistent with the fact that the nonlinear differential equation that relates the gauge-invariant and the gauge-dependent temporal-gauge fields — Eq. (90) — has a form very similar to the one that Gribov used to demonstrate the non-uniqueness of Coulomb-gauge fields — Eq. (98).

We are therefore led to conclude that the reason why the Gribov ambiguity does not arise when QCD is quantized in the temporal gauge — and most authors who discuss the quantization of QCD in the temporal gauge either do not mention the Gribov ambiguity or state that there is no Gribov ambiguity in the temporal gauge [28] — is that, in sharp contrast to the Coulomb gauge, quantization in the temporal gauge proceeds to completion without requiring the imposition of Gauss's law. The Gribov copies arise in the temporal-gauge formulation only after the theory has been quantized, and Gauss's law then is implemented and gauge-invariant fields are constructed. Our results are consistent with the conclusion that the Gribov ambiguity is a fundamental attribute of non-Abelian gauge theories. Gribov copies do not arise when a non-Abelian gauge theory is originally quantized in the temporal or other axial gauges, because, then, Gauss's law remains unimplemented. The Gribov ambiguity does manifest itself in these gauges, but only with the introduction of the gauge-invariant fields and the imposition of Gauss's law.

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Appendix

In this Appendix, we will evaluate the quantities $\mathcal{P}_{ui}$ and $\mathcal{R}_{vb}$ defined in Eq. (34), and show that they have the required properties.

For the case of $\mathcal{P}_{ui}$, we evaluate $V_{\mathcal{C}}\partial_i V_{\mathcal{C}}^{-1}$, where $V_{\mathcal{C}}$ is most conveniently represented as shown in Eq. (9), so that

$$\chi_i = V_{\mathcal{C}}\partial_i V_{\mathcal{C}}^{-1} = \exp\left(-ig\mathcal{Z}^\alpha(\mathbf{r})\tfrac{\tau^\alpha}{2}\right)\partial_i \exp\left(ig\mathcal{Z}^\alpha(\mathbf{r})\tfrac{\tau^\alpha}{2}\right). \tag{99}$$

For $\Phi^\alpha = g\mathcal{Z}^\alpha$ and $\Phi = \sqrt{\Phi^\alpha\Phi^\alpha}$, we obtain

$$\chi_i = \left\{\cos\left(\frac{\Phi}{2}\right) - i\frac{\tau^a\Phi^a}{\Phi}\sin\left(\frac{\Phi}{2}\right)\right\}\partial_i\left\{\cos\left(\frac{\Phi}{2}\right) + i\frac{\tau^a\Phi^a}{\Phi}\sin\left(\frac{\Phi}{2}\right)\right\}. \tag{100}$$

Eq. (34) then determines that

$$\mathcal{P}_{ai} = \partial_i \Phi^a + \left(\frac{\Phi^a \partial_i \Phi}{\Phi} - \partial_i \Phi^a \right) \left(1 - \frac{\sin \Phi}{\Phi} \right) + \epsilon^{abc} \Phi^b \partial_i \Phi^c \left(\frac{1 - \cos \Phi}{\Phi^2} \right). \tag{101}$$

Similarly, Eq. (34) also determines that $\mathcal{R}_{ba}$ is given by

$$\mathcal{R}_{ba} = \delta_{ab} \cos \Phi + \Phi^a \Phi^b \left(\frac{1 - \cos \Phi}{\Phi^2} \right) + \epsilon^{abc} \Phi^c \left(\frac{\sin \Phi}{\Phi} \right). \tag{102}$$

Inspection of Eq. (102) demonstrates that simultaneous substitution of $-\vec{\Phi}$ for $\vec{\Phi}$ and exchange of the subscripts a and b in $\mathcal{R}_{ba}$ leaves the expression on the right-hand side of this equation unchanged, thus proving Eq. (39).

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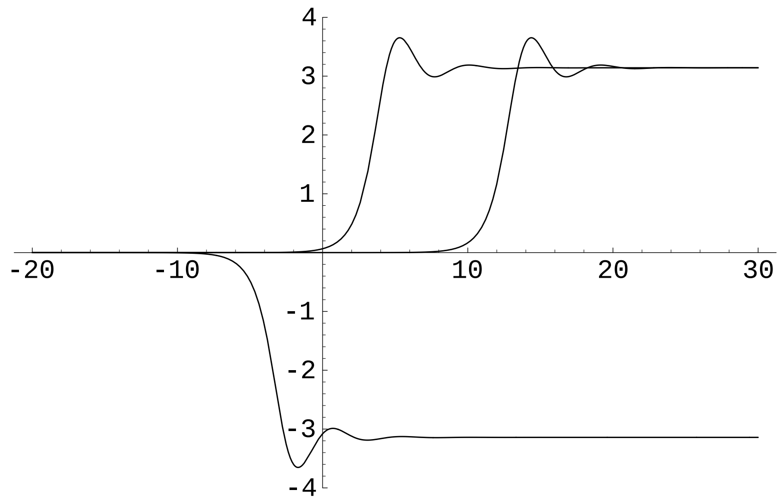

Figure 1:

Figure Caption

Three solutions of Eq. (94) with $\overline{\mathcal{N}}$ bounded in the interval $(-\infty \leq u \leq \infty)$. Of the two solutions with $\overline{\mathcal{N}} \geq 0$, one is obtained with $\overline{\mathcal{N}} = d\overline{\mathcal{N}}/du = 10^{-6}$ at $u = -11$ and the other with $\overline{\mathcal{N}} = d\overline{\mathcal{N}}/du = 10^{-10}$ at $u = -11$. The plot for the latter solution is identical to the plot of the former shifted to the right on the u axis. Both solutions approach $\overline{\mathcal{N}} = \pi$ as $u \rightarrow \infty$. The solution with $\overline{\mathcal{N}} \leq 0$ is obtained with $\overline{\mathcal{N}} = d\overline{\mathcal{N}}/du = -10^{-6}$ at $u = -18$. In this solution, $\overline{\mathcal{N}} \rightarrow -\pi$ as $u \rightarrow \infty$. The three solutions are precisely identical *modulo* shifts along the u axis and reflection in it.